Ringvorlesung Homework assignment

Please hand in your solutions before 1 August.

1. (Isometries of 2-dimensional hyperbolic space) In this question, we investigate the isometries of hyperbolic space, i.e. the manifold $\mathbb{H}^2 = \{(x,y) \mid y > 0\}$ with the metric $g = (dx^2+dy^2)/y^2$. A diffeomorphism ψ on an n-dimensional manifold is called "isometry" if $\psi^* g = g$, or in coordinates, if

$$g_{ij}(x) = g_{kl}(\psi(x)) \frac{\partial \psi_k(x)}{\partial x_i} \frac{\partial \psi_l(x)}{\partial x_j} \tag{1}$$

where $x = (x_1, ..., x_n)$ and $\psi(x) = (\psi_1(x), ..., \psi_n(x))$ are the coordinate components, and where a sum over k, l is understood (summation convention). For the case of hyperbolic space, let $x_1 = x, x_2 = y$, let $z = x + iy$, and define ψ through

$$\psi_1 = \mathrm{Re}\frac{az+b}{cz+d}\ , \quad \psi_2 = \mathrm{Im}\frac{az+b}{cz+d} \tag{2}$$

where a, b, c, d are given real numbers such that $ad - bc = 1$.

(o) Explain why the isometries of a given manifold (M, g) form a group under composition $\psi \circ \psi'$ of diffeomorphisms (with group inverse given by the inverse of the isometry viewed as a diffeomorphism).

(i) Verify that ψ as defined above is an isometry of hyperbolic space.

(ii) Denote by A the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

so that $\det A = 1$. Let $\psi^{(A)}$ be the isometry defined as above by the entries of A. Show that $\psi^{(A)} \circ \psi^{(A')} = \psi^{(AA')}$, where A' is similarly defined in terms of a', b', c', d', and where AA' is the product of matrices. What do we learn from this about the isometry group of $\mathbb{H}^2$?

2. (Killing vectors)

(i) A Killing vector field $\xi = \xi^i \partial_i$ for a metric g is a vector field such that the equation $\nabla_i \xi_j + \nabla_j \xi_i = 0$ is satisfied, where $\xi_j = g_{jk}\xi^k$. Here, ∇ is the Levi-Civita connection of g. Show that the following three vector fields are Killing vector fields in hyperbolic space, as defined in the previous question:

$$L_1 = \partial/\partial x, \quad L_2 = (y^2 - x^2)\ \partial/\partial x - 2xy\ \partial/\partial y, \quad L_3 = 2x\ \partial/\partial x + 2y\ \partial/\partial y\ . \tag{4}$$

(Recall that x, y refer to the coordinates of hyperbolic space, and that e.g. $\partial/\partial x$ stands for the vector field which in these coordinates has components $(1, 0)$, whereas $\partial/\partial y$ has components $(0, 1)$.) Argue that there cannot be any further, linearly independent, Killing vector fields.

(ii) Show that if ξ, η are two Killing fields on a manifold (M, g), then their commutator, $[\xi, \eta] \equiv (\xi^i \nabla_i \eta^j - \eta^i \nabla_i \xi^j)\partial_j$ also is Killing. Calculate the commutators of the vector fields in (i) and identify the Lie-algebra defined by these commutator relations.

(iii) For some metric on a manifold (M, g), suppose that ξ is Killing, and suppose that $\gamma(t)$ is a geodesic curve, meaning that $\dot{\gamma}^i \nabla_i \dot{\gamma}^j = 0$. Show that the function $f(t) = \dot{\gamma}^i(t)\xi_i(\gamma(t))$ is constant. Hint: you may start the calculation by $\dot{f} = \dot{\gamma}^i \nabla_i f = ...$ Thus, Killing vector fields give rise to constants of motion for the geodesic equation.

(iv) Now let $\gamma(t) = (x(t), y(t))$ be a curve in hyperbolic space, so that $\dot{\gamma} = (\dot{x}, \dot{y})$. Use the constants of motion f_1, f_2, f_3 obtained from L_1, L_2, L_3 to determine all possible geodesics in hyperbolic space.

3. (Friedmann equations) The Friedmann-Lemaîire-Robertson-Walker (FLRW)-metrics are of the form

$$g = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) .$$

The coordinates coordinates $(x^i) = (t, r, \theta, \phi)$ are such that $t \in (t_0, t_1)$, $0 < kr^2 < 1, \theta \in (0, \pi], \phi \in (0, 2\pi]$, and $i, j \in \{0, ..., 3\}$. The above "line element notation" is supposed to mean that the coordinate components are:

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2/(1 - kr^2)^{-1} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 \sin^2\theta \end{pmatrix} .$$

(The number $k \in \{\pm 1, 0\}$ determines the geometry of the spatial slices $\Sigma = \{x_0 = t\}$: For $k = 0$, we have a flat, for $k = 1$ a spherical, and for $k = -1$ a hyperbolic geometry.) $a(t)$ is a smooth, everywhere non-zero function on (t_0, t_1) called the "scale factor."

(i) Verify that, in those coordinates, the components R_{00}, R_{11} of the Ricci tensor are given by

$$R_{00} = -3\ddot{a}/a, \quad R_{11} = (a\ddot{a} + 2\dot{a}^2 + 2k)/(1 - kr^2) ,$$

where an overdot stands for $\partial/\partial t$.

(ii) Let the stress tensor be defined by $(T_i{}^j) = diag(-\rho, p, p, p)$, where ρ and p are smooth functions of t on (t_0, t_1) (note the position of the indices!). Derive the Friedmann equations from the 00 and 11-components of Einstein's equation (written in the form $R_{ij} = 8\pi G_N(T_{ij} - (1/2)g_{ij}T)$, with $T = T_{ij}g^{ij}$):

$$\begin{aligned} (\dot{a}/a)^2 &= (8\pi G_N/3)\ \rho - k/a^2 \\ \ddot{a}/a &= -(4\pi G_N/3)(\rho + 3p) . \end{aligned}$$

Argue that you will not get further independent equations from the other components.

(iii) Assume $\rho = -p$ and $k = 0$. Show that *deSitter spacetime* defined by

$$a(t) = e^{tH}$$

is a solution to both Friedmann equations, and determine the relation between H and ρ. $H = \dot{a}/a$ is called *Hubble constant.* (For a general FLRW metric, this will actually depend on time.)

4. (Gauss-Codacci equations and constraints) Let g be a Lorentz metric, and let Σ be a smooth embedded space like hyper surface with unit normal field n. As in the lectures, define a positive definite (after pull-back to Σ) metric on Σ by $h_{ij} = n_i n_j + g_{ij}$. We also write $h^i{}_j = g^{ik} h_{kj}$, and note that this defines, in each tangent space $T_p M, p \in \Sigma$, an orthogonal (relative to $g|_p$) projection onto $T_p\Sigma \subset T_p M$. Recall from the lectures that, given any tensor field t on M which, on Σ is equal to its projection by h (meaning $t_{i...j} = t_{k...l} h^k{}_i \dots h^l{}_j$) the derivative operator D intrinsic to Σ is defined by

$$D_m t_{i...j} = h^n{}_m h^k{}_i \dots h^l{}_j \nabla_n t_{k...l} \ .$$

Let the Riemann tensor of D denoted by $r_{jkl}{}^i$ (note that this is projected by h, as is evident from the defining relation

$$D_i D_j t_k - D_j D_i t_k = r_{ijk}{}^l t_l \ .$$

Let $K_{ij} = h^k{}_i h^l{}_j \nabla_k n_l$ be the extrinsic curvature of Σ.

(i) Demonstrate that $D_i h_{jk} = 0$, so that D is identified with the Levi-Civita connection of h.

(ii) Show that (with $R_{ijk}{}^l$ the Riemann tensor of g)

$$r_{ijk}{}^l = h_i{}^m h_j{}^n h_k{}^s h^l{}_t R_{mns}{}^t - K_{ik} K_j{}^l + K_{jk} K_i{}^l$$

(iii) Show that

$$D_i K^i{}_j - D_j K^i{}_i = R_i{}^k h_{kj} \ .$$

(Note that this is the "momentum constraint".)

(iv) Show that

$$2 G_{ij} n^i n^j = r + K_i{}^i K_j{}^j - K_{ij} K^{ij} \ ,$$

where r is the scalar curvature of h. (Note that this is the "Hamilton constraint".)

5. (Energy estimates and non-linear wave equation) Consider the scalar wave equation on $\mathbb{R}^n$

$$(-\partial_t^2 + \nabla^2) u = u^p$$

where p is a natural number and $\nabla = (\partial_1, \dots, \partial_{n-1})$, and where $u(t, x)$ is a function of the variables $t = x_0$ and $x = (x_1, \dots, x_{n-1})$. As in the lectures, we define the k-th energy norm $E_k[u](t)$ by the integral

$$E_k[u](t) = \frac{1}{2} \sum_{|\alpha| \le k} \int_{\mathbb{R}^{n-1}} [(\partial_t \partial^\alpha u)^2 + \sum_{j=1}^{n-1} (\partial_j \partial^\alpha u)^2 + (\partial^\alpha u)^2]_{x_0 = t}$$

We are using the usual multi-index notation ∂^α etc. with respect to the spatial coordinates $(x_1, \dots, x_{n-1})$, but not the time coordinate x_0. Assume that $u(t, x)$ is a smooth solution with smooth initial data $u(0, x) = U_0(x), \partial_t u(0, x) = U_1(x)$ of compact support. It is assumed that the solution is defined for $t \in [0, T)$.

(i) Derive the differential inequality:

$$\frac{d}{dt} E_k[u](t) \le C \|u^p\|_{H^k} E_k[u]^{1/2}(t) + E_k[u](t) \ .$$

(ii) Using the inequality ($k = \sum |\alpha_j|$)

$$\| \prod_{j=1}^{N} \partial^{\alpha_j} \phi_j \|_{L^2} \leq C \sum_{j=1}^{N} \|\nabla^k \phi_j\|_{L^2} \prod_{j \neq i} \|\phi_i\|_{L^\infty} \ ,$$

for $\phi_j \in C_0^\infty(\mathbb{R}^{n-1})$, show that

$$\frac{d}{dt} E_k[u](t) \leq C(\|u(t,\ .\ )\|_{L^\infty}) E_k[u](t) \ .$$

for a suitable function $C(\ .\ ) \geq 0$.

(iii) Use the Gronwall lemma to obtain from (ii) a bound for $E_k[u](t)$ of the form

$$E_k(t) \leq E_k(0) \exp\left(\int_0^t C(\|u(t',\ .\ )\|_{L^\infty}) dt' \right) \ .$$

Argue that if $u(t,\ .\ )$ remains uniformly bounded almost everywhere for $t < T$, then the solution u can be continued beyond T (here you must use the local existence theorem. What do you know about the Sobolev norms $u, \partial_t u$ (in x) in the time interval $[0, T)$ from the above inequality?). Thus, we conclude that the solution either has to blow up somewhere or it can be extended.

(iv) Repeat the kind of estimations for the wave equation

$$(-\partial_t^2 + \nabla^2) u = v^p$$

where v is given. Show that

$$\frac{d}{dt} E_k[u](t) \leq C(\|v(t,\ .\ )\|_{L^\infty}) \|v(t,\ .\ )\|_{H^k} E_k[u]^{1/2}(t) + E_k[u](t) \ .$$

(v) As in the lecture, in order to find a solution u to the non-linear wave equation $(-\partial_t^2 + \nabla^2) u = u^p$ (locally in time), one sets up a Picard type iteration

$$(-\partial_t^2 + \nabla^2) u_{j+1} = u_j^p$$

We assume inductively that $\|u_j(t,\ .\ )\|_{H^k} \leq M$ for some M and $t \in [0, T)$. Using the Sobolev embedding theorem, argue that u is bounded in L^∞ for sufficiently large k. Next, using (iv), show that $\|u_{j+1}(t,\ .\ )\|_{H^k} \leq M$ holds for $t \in [0, T)$, provided that M was chosen sufficiently large in terms of U_0, U_1 and provided that T is chosen sufficiently small (independently of j).