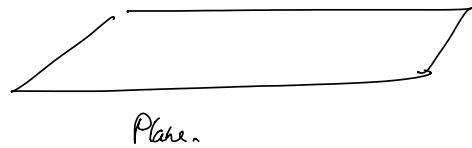
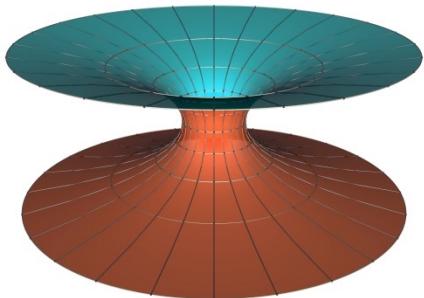
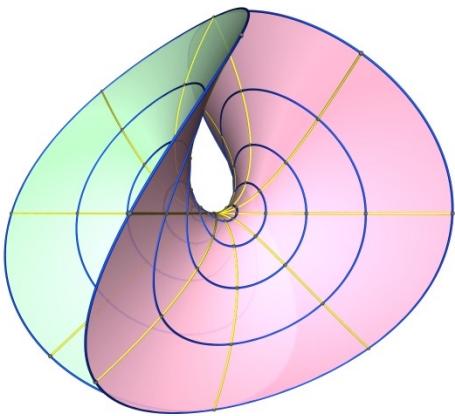


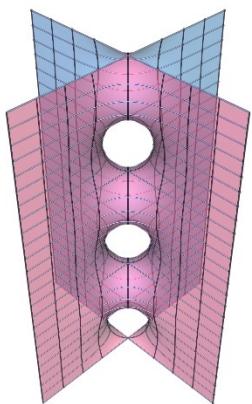
Catenoid

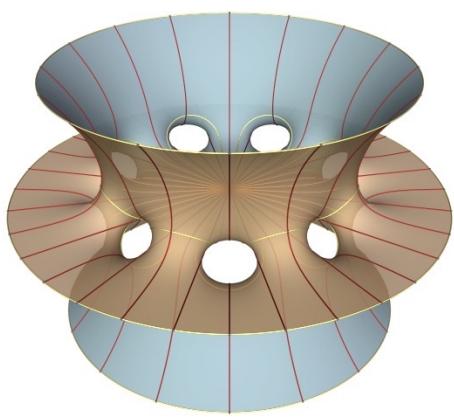


Enneper

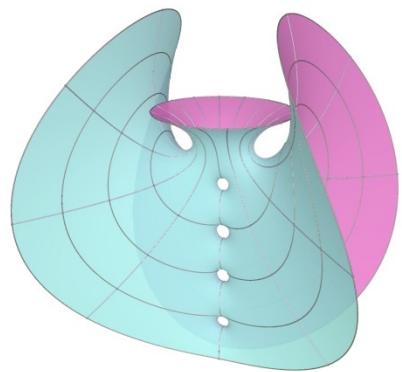


Singly periodic Scherk



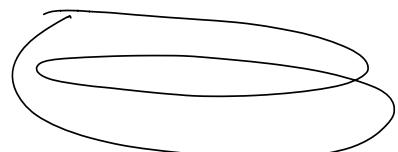
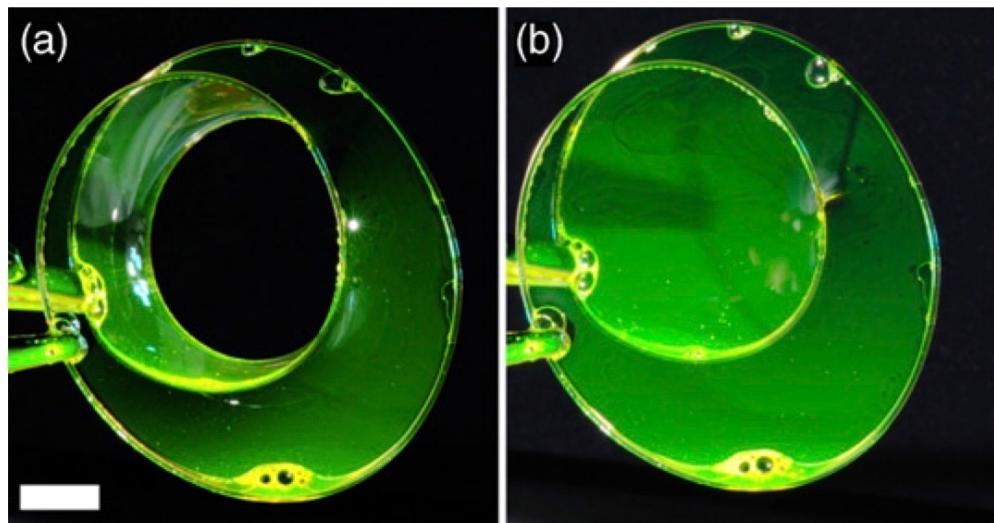


Costa - Hoffman - Necks II



Catenoid + Enneper of genus 5

Images from the "Minimal Surface Archive" by Matthias Weber.



Classical minimal surfaces and their genus.

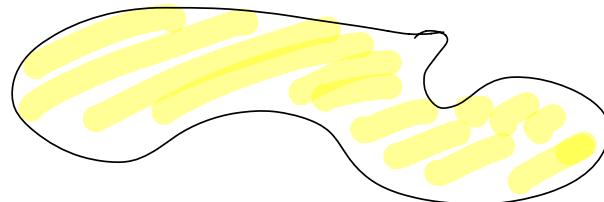
Motivation:

Problem 1: (Plateau problem)

Given a Jordan curve $\Gamma \subseteq \mathbb{R}^3 (\mathbb{R}^N)$ find a 2 dimensional surface $\text{spanning } \Gamma$ with least area.



?



What do we consider
to be a surface

- Questions:
- (a) What do we consider to be a surface ?
 - ↳ - parametric approach (in 2D)
 - GFT
 - varifolds / currents / sets of finite perimeter
 - rectifiable sets
 - (b) How to we understand spanning ?
 - Homology theory (currents)
 - set theoretic (Direct approach to plateau's problem)
 - (c) What do we consider to be the "least" area ?
 - current (mass, p-mass ...)
 - Hausdorff measure.
 - (d) Can / How do we find such a surface ?
 - Calculus of variations.
 - (e) What are its properties?
 - (e1) from an analytic point of view :
 - regularity (Smoothness / Branch points)
 - uniqueness
 - $\xrightarrow{\text{parametric}} \text{harmonic functions}$
 - $\xrightarrow{\text{currents}} \text{c-regularity}$
 - (e2) from a geometric point of view :
 - Embeddability \longrightarrow "easy" in setting of currents
"hard" in the parametric approach.
 - Genus

↓

"easy" in the parametric approach
 "hard" in the setting of currents.

I will focus mainly on 2-dimensional minimal surfaces!

Today: B. White : lectures on minimal surfaces.

1.1 The first variation of the area + consequences

We assume Ω m-dimensional surface (properly immersed) in \mathbb{R}^N

and at least C^1 -regular.

i.e. $\exists \{\phi_i\}_i$ of C^1 -maps $\phi_i: U_i \rightarrow \mathbb{R}^N$
 s.t. ϕ_i is a local parametrisation. \mathbb{R}^m of Ω
 and $\Omega \subseteq \bigcup_i \phi_i(U_i)$

Dohruck: "least" area \Rightarrow necessary condition $\left. \frac{d}{dt} \right|_{t=0} \text{area}(\Omega_t) = 0$

Theorem 1.1: (1. variation).

Let Ω be as above, $\phi_t: \Omega \rightarrow \mathbb{R}^N$ a one parameter C^1 -family
 of immersions s.t. $\phi_0(p) = p \quad \forall p \in \Omega$ and $\phi_t(p) = p \quad \forall p$ outside
 a compact set, and $X = \left. \frac{d}{dt} \right|_{t=0} \phi_t$ the initial velocity vectorfield.

then

$$\left. \frac{d}{dt} \right|_{t=0} \text{area}(\phi_t(\Omega)) = \int_{\Omega} \text{div}_{\Omega}(X) \, dS \quad (1)$$

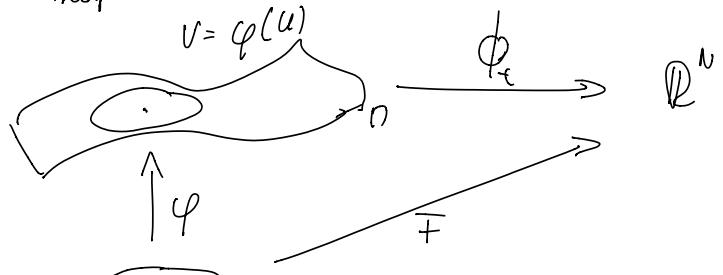
generalized divergence theorem. \hookrightarrow

$$= \int_{\partial\Omega} X \cdot \nu_{\Omega} \, ds - \int_{\Omega} H \cdot X \, dS \quad (2)$$

mean curvature of Ω .

where $\text{div}_{\Omega}(X) = \sum_{i=1}^n \langle \tau_i, D_{\tau_i} X \rangle$ where $\{\tau_i\}_{i=1}^n$ is an ONB of
 $T\Omega$ derivative in \mathbb{R}^N (not the Levi-Civita connection on Ω)

Proof:



$$F = \phi_t \circ \varphi : [-\varepsilon, \varepsilon] \times U \rightarrow \mathbb{R}^N$$

$$\begin{aligned} \text{area}(\phi_t(V)) &= \text{area}(\bar{F}_t(u)) \\ &= \int_U \bar{F} \, dx \\ &= \int_U \sqrt{\det(g)} \, dx \end{aligned}$$

$$\begin{aligned} \bar{F}^2 &= \det(\partial_t^\top \partial F_t) \\ &= \det(g) \end{aligned}$$

$$g = \bar{F}^* S_{\mathbb{R}^N} \quad g_{ij} = \langle \partial_i F, \partial_j F \rangle$$

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \text{area}(\bar{F}_t(u)) &= \int_U \frac{d}{dt} \Big|_{t=0} \bar{F} \Big|_{t=0=h} \, dx \\ &= \int_U \underbrace{\frac{d}{dt} \bar{F}}_{\sqrt{\det(g)}} \Big|_{t=0} \, \sqrt{\det(g)} \, dx \\ &= \int_V h \, dS \end{aligned}$$

We need to calculate h :

$$g_{ij}(t) = \langle \partial_i F_t, \partial_j F_t \rangle$$

$$\frac{d}{dt} g_{ij}(t) = \langle \partial_i \partial_t F_t, \partial_j F_t \rangle + \langle \partial_i F_t, \partial_j \partial_t F_t \rangle$$

$$\text{at } t=0 \Rightarrow \dot{g}_{ij}(0) = \langle \partial_i X, \partial_j F \rangle + \langle \partial_i F, \partial_j X \rangle$$

$$\Rightarrow \det(g_{ij}(t)) = \det(g_{ij}(0)) \left(1 + t \cdot \text{tr}(g^{ij}(0) \dot{g}_{ij}(0)) + o(t) \right)$$

because

$$\begin{aligned} g_{ij}(t) &= g_{ij}(0) + t \dot{g}_{ij}(0) + o(t) \\ &= g_{ik}(0) \left(S_{ij}^k + t g^{kl}(0) \dot{g}_{kl}(0) + o(t) \right) \end{aligned}$$

$$\Rightarrow \frac{d}{dt} \overline{\det(g_{ij}(t))}^T = \frac{1}{2} \overline{\det(g_{ij}(0))}^T g^{kl}(0) g_{kl}(0)$$

$$= \overline{\det(g_{ij}(0))}^T g^{kl}(0) \underbrace{\langle \partial_k F, \partial_l X \rangle}_{D_F X}$$

choose φ s.t. is a point p on Ω

we have $\langle \partial_i F, \partial_j F \rangle = \delta_{ij}$ \Rightarrow in this point.

$$\frac{d}{dt} \overline{\det(g_{ij}(t))}^T \Big|_{t=0} = \langle \partial_i F, D_{\partial_i F} X \rangle = \operatorname{div}_\Omega(X) \Rightarrow (2)$$

"Generalized divergence theorem"

$$\operatorname{div}_\Omega(X) = \operatorname{div}_\Omega(\underbrace{X^\top}_{\text{orthogonal part.}}) + \operatorname{div}_\Omega(\underbrace{X^\perp}_{\text{tangential part}})$$

Let $\{\tau_e\}_e$ be a local orthonormal frame

$$\operatorname{div}_\Omega(X^\top)(p) = \sum_e \langle \tau_e, D_{\tau_e} X^\top \rangle = \sum_e \langle \tau_e, D_{\tau_e} X^\perp \rangle$$

$(= d(X^\top \downarrow dS))$ Celi-Chinta connection.

$$\operatorname{div}_\Omega(X^\perp)(p) = \sum_e \langle \tau_e, D_{\tau_e} X^\perp \rangle = - \sum_e \langle (D_{\tau_e} \tau_e)^T, X \rangle$$

$$= - \sum_e \langle A(\tau_e, \tau_e), X \rangle = - \langle H, X \rangle$$

$$\Rightarrow \int_{\Omega} \operatorname{div}_\Omega(X) dS = \int_{\Omega} \operatorname{div}_\Omega(X^\top) dS + \int_{\Omega} \operatorname{div}_\Omega(X^\perp) dS$$

$$= \int_{\partial\Omega} X \cdot \nu dS - \int_{\Omega} H \cdot X dS$$

□

Definition 1.2: A C^2 -regular m -dimensional submanifold of \mathbb{R}^N is called minimal if $H = 0$ $\Rightarrow \frac{d}{dt} \operatorname{area}(\phi_t(\Omega)) = 0 \quad \forall t$ with $X = 0$ on $\partial\Omega$.

Corollary 1.2: If minimal and compact

$$\text{in area } (\Omega) = \int_{\partial\Omega} (x - q) \cdot \nu \, ds \quad (3)$$

for $q \in \mathbb{R}^N$ arbitrary.

Proof: Take the vectorfield $X = (x - q)$ then $\operatorname{div}_{\Omega}(X) = m$
hence (1) & (2) \Rightarrow (3) \square

Corollary 1.4: The coordinate functions are harmonic on Ω :

$$(4) \quad \Delta_g x^i = 0 \quad \text{on } \Omega$$

$$\begin{aligned} \Delta_g x^i &= \operatorname{div}_{\Omega}(\nabla x^i) = \operatorname{div}_{\Omega}((Dx^i)^T) \\ &= \operatorname{div}_{\Omega}(Dx^i - (Dx^i)^{\perp}) \\ &= \operatorname{div}_{\Omega}(e_i) + H \cdot Dx^i = 0 \end{aligned}$$

Remark: Using the maximum principle to derive that.

$\Omega \subseteq \operatorname{conv}(\partial\Omega) \Rightarrow$ there is no compact minimal surface without boundary in \mathbb{R}^N .

1.2 Monotonicity

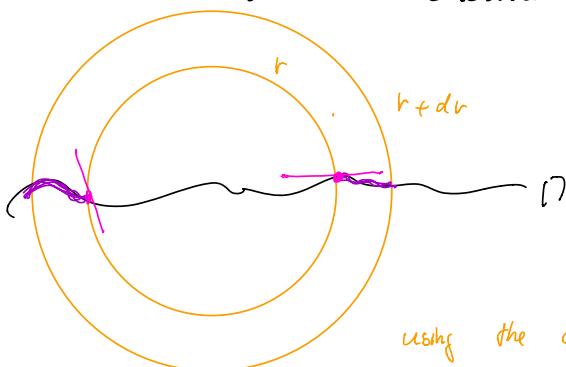
Lemma 1.5: Let Ω be minimal and $p \notin \partial\Omega$
then $r \mapsto \Theta(p, r) = \Theta(\Omega, p, r) = \frac{\text{area}(\Omega \cap B_r(p))}{\omega_m r^m}$
is monotone increasing $[0, \operatorname{dist}(p, \partial\Omega)]$ i.e.

$$\frac{d}{dr} \Theta(p, r) \geq 0$$

"=" iff $\partial B_r(p)$ intersects Ω orthogonal

Proof: We assume that $p=0$ and set $A(r) = \text{area}(\Omega \cap B_r)$

We want to calculate $A'(r)$



$$A(r+dr) - A(r)$$

$$= \text{area}(\Omega \cap B_{r+dr} \setminus B_r)$$

$$\gtrsim \text{area}(\Omega \cap \partial B_r) dr$$

$$\Rightarrow A'(r) \geq \text{area}(\Omega \cap \partial B_r)$$

using the co-area formula:

$$A'(r) = \int_{\partial B_r \cap \Omega} \frac{|x|}{|x|^2} ds \quad (4)$$

Corollary 2.3

$$\Rightarrow m A(r) = m \text{area}(\Omega \cap B_r) = \int_{\Omega \cap \partial B_r} x \cdot \nu \ ds \leq \int_{\Omega \cap \partial B_r} |x| \ ds = r \text{area}(\Omega \cap \partial B_r) \leq r A'(r)$$

$$\Rightarrow m A(r) \leq r A'(r)$$

$$(r^{-m} A(r))' \geq 0$$

If one uses (4) $\Rightarrow \frac{A(R)}{R^m} - \frac{A(r)}{r^m} = \int_{\Omega \cap B_R \setminus B_r} \frac{|x|^{1/2}}{|x|^{m+2}} ds$ \square

Definition: 2.6: For each $p \in \Omega \setminus \partial \Omega$ we define

$$\Theta(p) = \lim_{\substack{r \rightarrow 0 \\ r > 0}} \Theta(p, r)$$

Remark: If Ω is C^1 then $\Theta(p) = \#\{\text{sheets passing through } p\}$

Q: Can we make sense of $\lim_{r \nearrow \infty} \Theta(p, r)$

Ans: If $\partial \Omega = \emptyset$ then, yes since $\Theta(p, r)$ is monot.

and since $B_r(p) \subseteq B_{r+|q-p|}(q) \nvdash p, q$

$$\Rightarrow \Theta(\Sigma) = \lim_{r \rightarrow \infty} \Theta(p, r) \text{ does not depend on } p.$$

Corollary 7.7: There is no compact minimal surface in \mathbb{R}^N without boundary.

Proof: Assume Ω is minimal and compact. Take $p \in \bar{\Omega}$

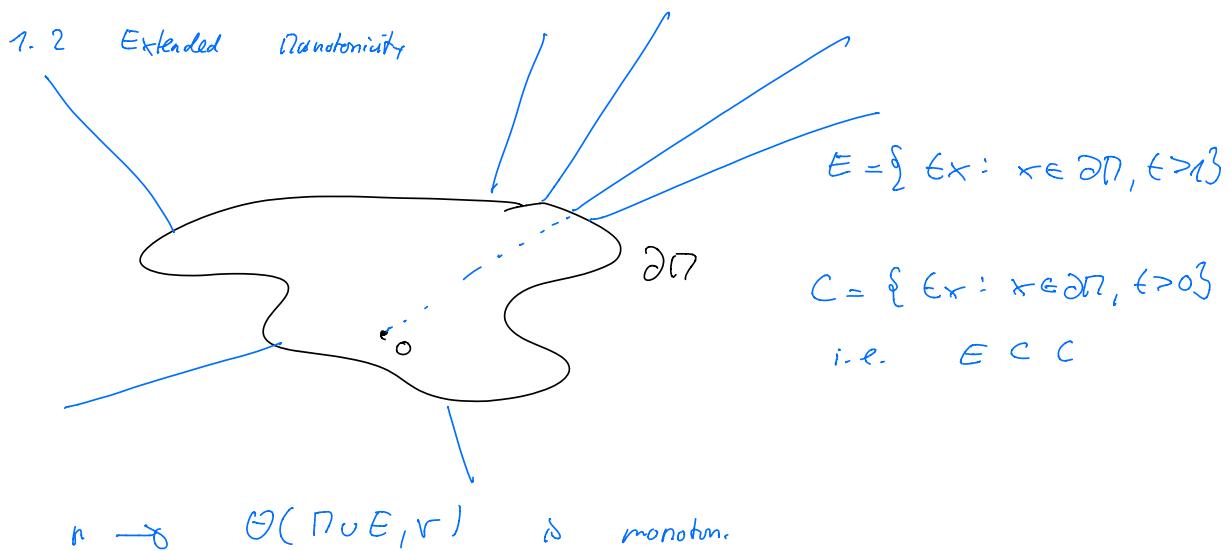
$$\Rightarrow 1 \leq \lim_{r \rightarrow 0} \Theta(p, r) \leq \lim_{r \rightarrow \infty} \Theta(p, r) = 0 \quad \leftarrow \text{Q.E.D.}$$

Corollary 7.8: Let Ω be properly immersed then $\Theta(\Omega) \geq 1$
and $\Theta(\Omega) = 1$ iff Ω is a single plane.

Proof: Pick $p \in \Omega$ so $1 \leq \Theta(p, r) \leq \Theta(\Omega) = 1$
 $\Theta'_{\rho}(p, r) = 0 \quad \forall r \Rightarrow \partial B_r(p) \perp \Omega \Rightarrow \Omega$ is a cone.

$\Rightarrow \Omega$ is smooth so it's a collection of planes $\Rightarrow \Omega$ is a single plane.

17



then $\lim_{r \rightarrow \infty} \Theta(\Omega \cup E, r) = \Theta(C) = \Theta(C, 0, r)$ for

$$= \Theta((\partial\Omega)^*)$$

$(\partial\Omega)^*$ is the projection of $\partial\Omega$ on S^{N-1}

Theorem 1.9 (Embodies / no branchpoints)

Let D be 2-dimensional s.t.

$$\int_{\partial D} |k| < 4\pi$$

then D is embedded (and does not have branchpoints).

Proof:

$$\text{Length}(\gamma^*) \leq \int_{\gamma} |k|$$

\Rightarrow Suppose D is not embedded $\exists p \in D$ where $\#\{\text{sheets}\} \geq 2$

$$\begin{aligned} 2 &\leq G(p) \leq \Theta(D \cup E, p) \leq \Theta(C) \\ &= \frac{1}{2\pi} \cdot \text{Length}(\gamma^*) < 2 \end{aligned}$$

□

Corollary 1.9: (Fary, Nöller)

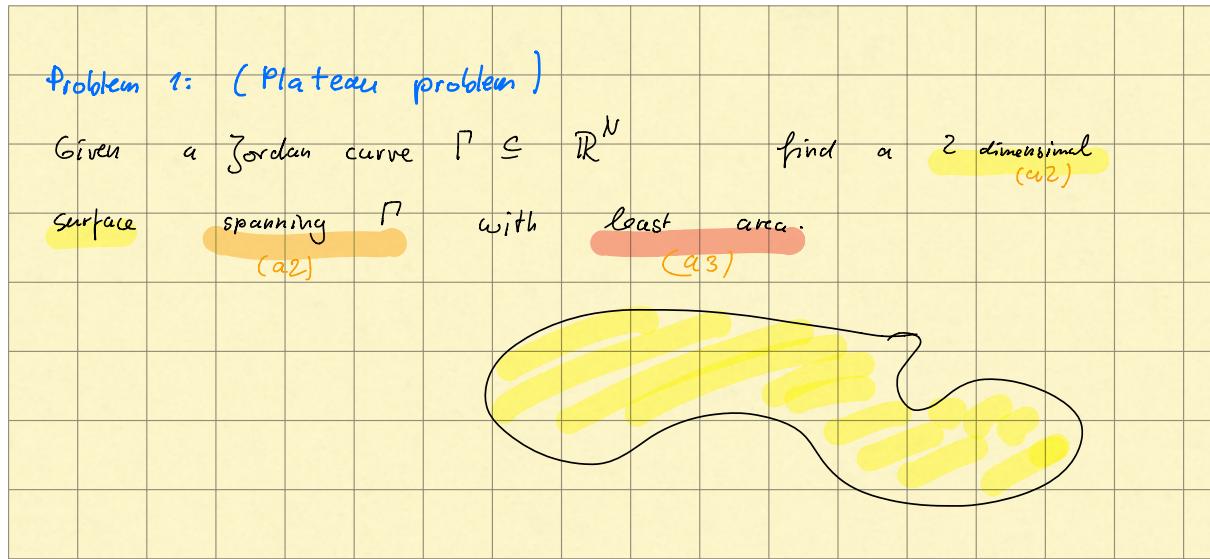
If γ is a Jordan curve in \mathbb{R}^3 with $\int_{\gamma} |k| < 4\pi$
then γ is unknotted.

Proof: Next time we will show $\exists (!)$ D minimal disk with
 $\partial D = \gamma \stackrel{\text{Theorem 1.73}}{\Rightarrow} D$ is embedded.

$\Rightarrow D$ gives homotopy between γ and the unknot.

□

2. Lecture: The variational approach by Douglas-Rado.



Assumptions for today:

(a1) Γ is a smooth Jordan curve.

(a2) "2 dimensional surface" :

$X: D \subseteq \mathbb{C} \rightarrow \mathbb{R}^N$ s.t. $X: \partial D \rightarrow \Gamma$ is monotonic parametrization.

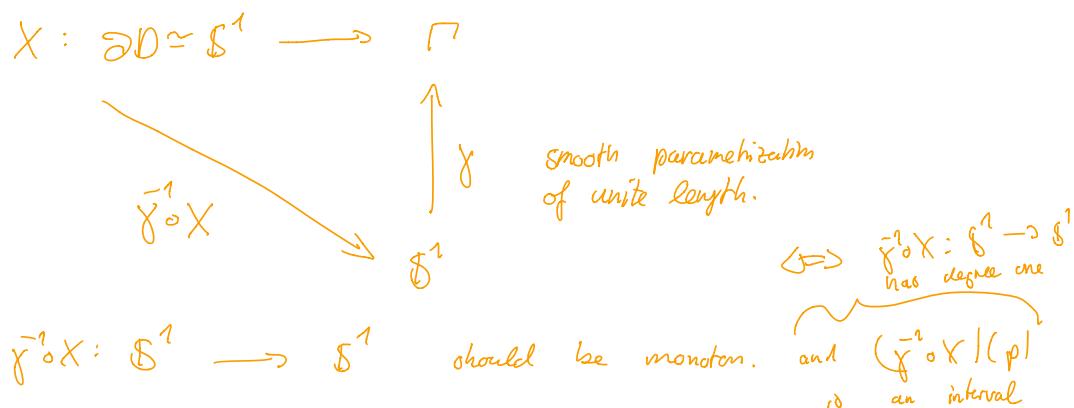
(a3) $A(X) := \text{area}(X) = \int_D JX \, dx$
 where $JX(z) = \sqrt{|\partial_1 X|^2 |\partial_2 X|^2 - \langle \partial_1 X, \partial_2 X \rangle^2}$

"least area" $A(X) \leq A(Y) \quad \forall Y \text{ as in (a2)}$ (D)

Some comments

(a2) X should be locally Lipschitz

X weakly monotonic means.



We will denote this family of maps $C(\Gamma)$

(a3) We measure the mapping area because the area formula we have

$$\int_D \mathcal{J}X \, dx = \int_{X(D)} \# \{ X^{-1}(y) \} \, d\mathcal{H}^2(y)$$

Question for today: How to find X satisfying (17) ?

(D) Direct method of the calculus of variations:

How to find x_0 with $f(x_0) = \inf f$?

(i) take a minimizing sequence (x_n) (always exists)

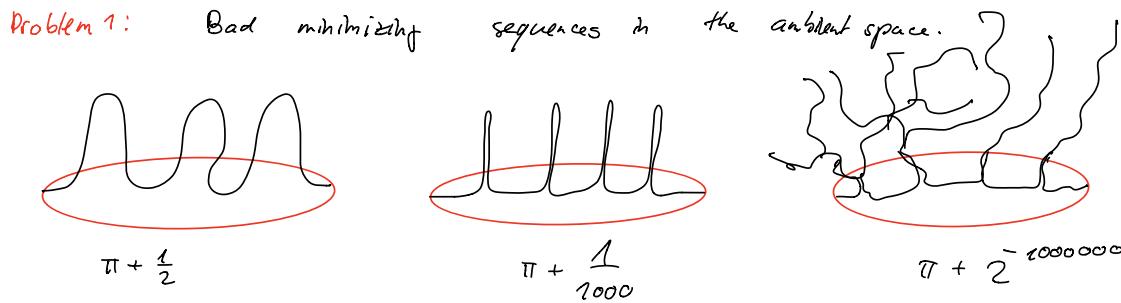
(ii) show the sequence has a converging subsequence $(x_{n_k})_k$

(iii) $\liminf_{k \rightarrow \infty} f(x_{n_k}) \geq f(x_0)$

choose a weak topology

otherwise you're restricting yourself to finite dimensional problems.

In our case $f(X) = A(X)$



Problem 2: bad sequence on the domain.

Diffeomorphism invariance of the area

$$A(X) = A(X \circ \varphi) \quad \forall \varphi: D \rightarrow D \text{ diffeomorphism.}$$

Take $\varphi_n(z) = (1 - \frac{z}{n}) |z|^{2n} z + \frac{z}{n} z$
 $X \circ \varphi_n(z) \rightarrow X(0) \quad (n \rightarrow \infty) \quad \forall z \in D$

but $A(X(0)) = 0$

Start again with the necessary condition:

Last time: D is minimal $\Rightarrow \Delta_g x^i = \omega_N \cdot (\nabla x^i) = \omega_N (e_i - e_i^{-1}) = H \cdot e_i = 0$

if $X: D \rightarrow \mathbb{R}^N$ is minimal $\Rightarrow \Delta_g X = 0$

① in 2D Δ_g is conformal invariant.

Corollary 2.1: Let $X: D \rightarrow \mathbb{R}^N \subset \mathbb{C}^2$ with

(c) $|\partial_1 X|^2 = |\partial_2 X|^2 \quad \& \quad \langle \partial_1 X, \partial_2 X \rangle = 0$ (almost conformal)
 since we do not ask for $|\partial X| \neq 0$

then X is minimal iff $\Delta X = 0$

Consequence: if X is minimal satisfying (c) $\Rightarrow X$ is analytic in D

PDE Solution to the Plateau Problem:

$X: D \rightarrow \mathbb{R}^N$ satisfying (S1) X satisfying (C)

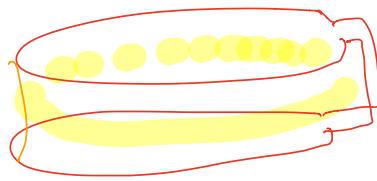
$$(S2) \quad \Delta X = 0$$

(S3) $X: \partial D \rightarrow \mathbb{R}$ is monotone and of degree 1.

(like the Euler-Lagrange Equation to the Plateau Problem)

Q: Does (S1) - (S3) imply (D)?

Answer: No



Idea of Douglas & Rado (1930/31)

Instead of A consider $E(X) := \frac{1}{2} \int_D |DX|^2 (= \|X\|_H^2)$

because (S2) is the Euler-Lagrange equation to E .

Q: Can we apply (D) to E ?

Answer: kind of

assume $g \in C^1$

(a) If we consider $C_g^1(D) := \{X \in C^1(D; \mathbb{R}^N) : X = g \text{ on } \partial D\}$

$\Rightarrow \overline{C_g^1(D)}^{H^1_H} =: H_g^1 C^1_{\mathbb{R}}$ is Hilbertspace.
affine subset.

• norm is lower-semicontinuous.

• bounded sequences in H_g^1 are precompact in the weak topology

\Rightarrow (D) works for g fixed!

(B) Problem 3: Möbius invariance of E :

$$E(X \circ m) = E(X) \quad \text{for all } m: D \rightarrow D \text{ Möbius}$$

take $m_k(z) = \frac{z - \omega_k}{1 - \bar{\omega}_k z}$ with $|\omega_k| \rightarrow 1$

then $X \circ m_k \rightarrow X(1)$ on D .

$$\Rightarrow (D) \text{ does not work.} \quad \overline{C(\Gamma)}^{H^1} = H(\Gamma) \subseteq H^1$$

Strategy: Step 1: $\inf A(X) \leq \inf E(X)$

Step 2: "break" the Möbius invariance of E on $C(\Gamma)$

Step 3: $\inf A(X) = \inf E(X)$.

Step 1: had been shown by Riesz before.

Lemma 2.2: If $X \in C(\Gamma)$ we have $A(X) \leq E(X)$ with " $=$ "

iff X is conformal.

Proof:

$$\begin{aligned} JX &= \sqrt{|\partial_1 X|^2 |\partial_2 X|^2 - \langle \partial_1 X, \partial_2 X \rangle^2} \\ &\stackrel{"="}{\Leftrightarrow} |\partial_1 X| = |\partial_2 X| \\ &\stackrel{?}{\leq} |\partial_1 X| |\partial_2 X| \leq \frac{1}{2} (|\partial_1 X|^2 + |\partial_2 X|^2) = \frac{1}{2} |DX|^2 \\ &\stackrel{"="}{\Leftrightarrow} \langle \partial_1 X, \partial_2 X \rangle = 0 \end{aligned}$$

$$\Rightarrow \inf A(X) \leq \inf E(X)$$

Lemma 2.2: Variational characterization of (S1) & (S2)

Let $X \in H(\Gamma)$

(3)

then (i) $\Delta X = 0$ in $D \Leftrightarrow \left. \frac{d}{dt} \right|_{t=0} E(X + t\varphi) = 0 \quad \forall \varphi \in C_c^1(D; \mathbb{R}^N)$
 $\Leftrightarrow E(X) \leq E(Y) \quad \forall Y = X \text{ on } \partial D$

(ii) X is almost conformal $\Leftrightarrow \left. \frac{d}{dt} \right|_{t=0} E(X \circ \phi_t^{-1}) = 0 \quad \forall \phi_t: D \rightarrow \mathbb{C}$
 $\text{C}^1\text{-family}$

Proof: (i) (3) Let $Y = X + t\varphi$

$$\begin{aligned}
 E(Y) - E(X) &= \int_0^t \frac{1}{2} \|DX + tD\varphi\|^2 - \frac{1}{2} \|DX\|^2 \\
 &= t \int_0^t DX \cdot D\varphi + \frac{t^2}{2} \int_0^t \|D\varphi\|^2 \\
 &\stackrel{\varphi \in C^2}{=} t \underbrace{\int_0^t (\Delta X) \varphi}_{X \text{ is } hC^2} + \underbrace{\frac{t^2}{2} \int_0^t \|D\varphi\|^2}_{\geq 0} \\
 &= 0 \quad \text{iff} \quad \Delta X = 0
 \end{aligned}$$

(ii) $D(X \circ \phi_t^{-1}) = D(X(\phi_t^{-1})) D\phi_t^{-1}$ and let $\varphi = \frac{d}{dt}|_{t=0} \phi_t \in C^1(0, \mathbb{C})$

$$\begin{aligned}
 E(X \circ \phi_t^{-1}) &= \frac{1}{2} \int_0^t \|DX(\phi_t^{-1}) D\phi_t^{-1}\|^2 \\
 &\stackrel{\phi_t(0)}{=} t + \epsilon \text{tr}(D\varphi) + o(t) \\
 &= \frac{1}{2} \int_0^t \|DX(z) \underbrace{D\phi_t^{-1} \circ (\phi_t)}_{\text{id} - t D\varphi + o(t)}\|^2 \det(D\phi_t) \\
 &= E(X) + t \int_0^t \frac{1}{2} \|DX\|^2 dz(z) - DX \cdot (D\varphi)^T DX + o(t) \\
 \varphi &= (\varphi^1, \varphi^2) \\
 \frac{d}{dt} E(X) &= \int_0^t \frac{1}{2} \partial_i \varphi^i \|DX\|^2 - \partial_i X \cdot \partial_j X \partial_i \varphi^j \\
 &= \int_0^t \partial_1 \varphi^1 \left(-\frac{1}{2} |\partial_1 X|^2 + \frac{1}{2} |\partial_2 X|^2 \right) - \partial_2 \varphi^1 \partial_2 X \cdot \partial_1 X \\
 &\quad + \partial_2 \varphi^2 \left(\frac{1}{2} |\partial_1 X|^2 - \frac{1}{2} |\partial_2 X|^2 \right) - \partial_1 \varphi^2 \partial_1 X \cdot \partial_2 X \\
 &= \int_0^t (-\partial_1 \varphi^1 u + \partial_2 \varphi^1 v) - (\partial_2 \varphi^2 u + \partial_1 \varphi^2 v)
 \end{aligned}$$

$$\text{where } u = \frac{1}{2} |\partial_1 X|^2 - \frac{1}{2} |\partial_2 X|^2 \quad v = -\partial_1 X \cdot \partial_2 X$$

$$= \int_0^1 \operatorname{curl}(\varphi) u + \operatorname{div}(\varphi) v$$

Let f be C^1 and $\Delta \varphi = f$
 Pick $\varphi = \nabla \psi$ $\varphi = \nabla^\perp \psi$

$$(I) \text{ implies } \left. \frac{d}{dt} E(X(id + t \nabla \psi)) \right|_{t=0} = \int \Delta \varphi v = \int f v$$

$$(II) \quad \left. \frac{d}{dt} E(X(id + t \nabla^\perp \psi)) \right|_{t=0} = \int \Delta \varphi u = \int f u$$

2. remaining problems:

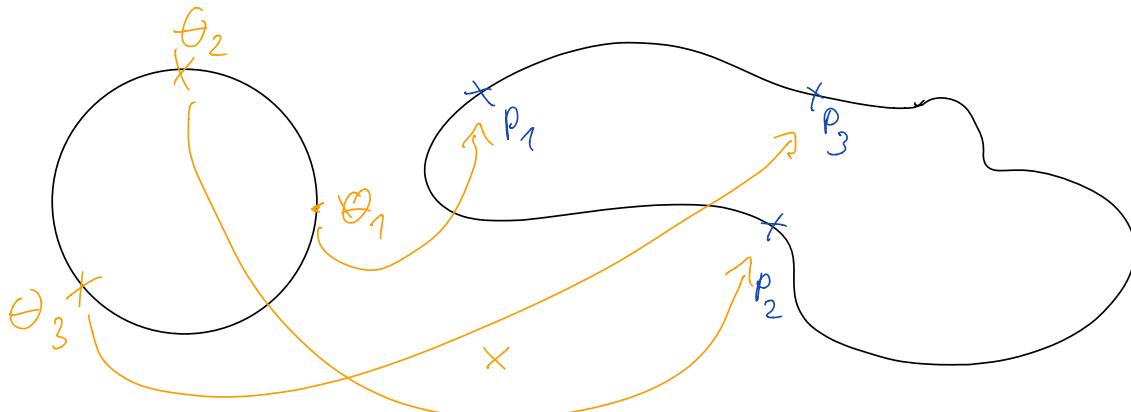
(a) H^* obvious invariance $\Rightarrow H^*(P)$

(b) bounded sequence in $H(P)$ are precompact.

(a) 3 Point condition:

$$\mathcal{M} = \left\{ e^{i\phi} \frac{z-\omega}{z-\bar{\omega}z} : \omega \in D, \phi \in \mathbb{R} \right\}$$

3-parameter family.



$$H^*(P) = \{ X \in H^1 : X : \partial D \rightarrow P \text{ satisfy (c)} \\ X(e^{i\theta_j}) = p_j \text{ for } j=1,2,3 \}$$

Note that $H^*(P) \subset H(P)$

$$\Rightarrow \inf_{H^*(P)} E \geq \inf_{H(P)} E$$

if $X \in H(P)$ $\exists m \in M$ s.t. $X_m \in H^*(P)$

$$\Rightarrow E(X) = E(X_m)$$

(b) We need to show that (c) is preserved.

This is a consequence of

Lemma 2.3: (Courant-Lebesgue)

Let $X \in H^1(\Omega; \mathbb{R}^N)$ and $D_{R^2} \cap \Omega \neq \emptyset$

(i) for a.e. $0 \leq r \leq R^{\frac{1}{2}}$

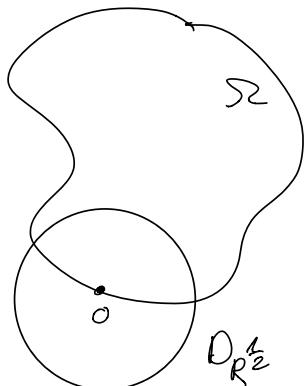
$$\Theta \mapsto X(re^{i\Theta}) = X_r(\Theta)$$

Θ absolutely continuous.

$$(ii) \int_0^{R^{\frac{1}{2}}} \frac{\text{length}(X_r)^2}{r} dr \leq \int_{D_{R^2} \cap \Omega} |DX|^2 = 2E(X; \Omega \cap D_R)$$

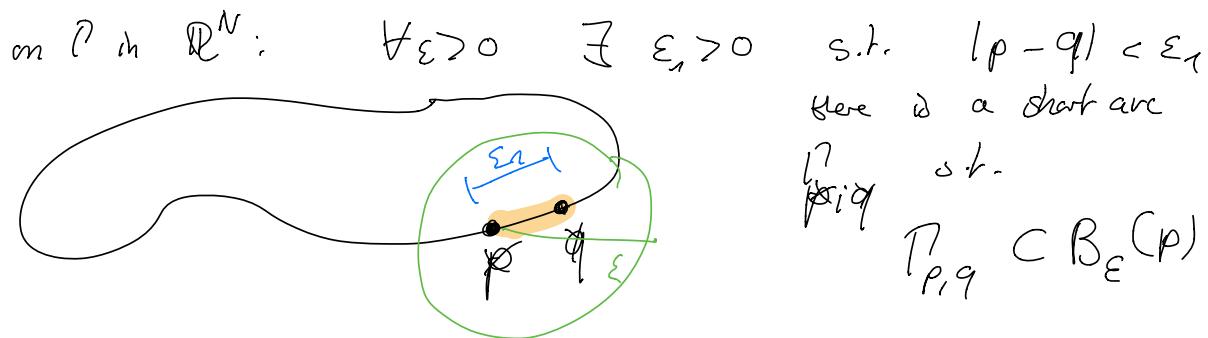
(iii) In particular $\forall R < 1 \exists R < \rho < R^{\frac{1}{2}}$ s.t.

$$\text{length}(X_\rho)^2 \leq \frac{8E}{|\ln(R)|}$$

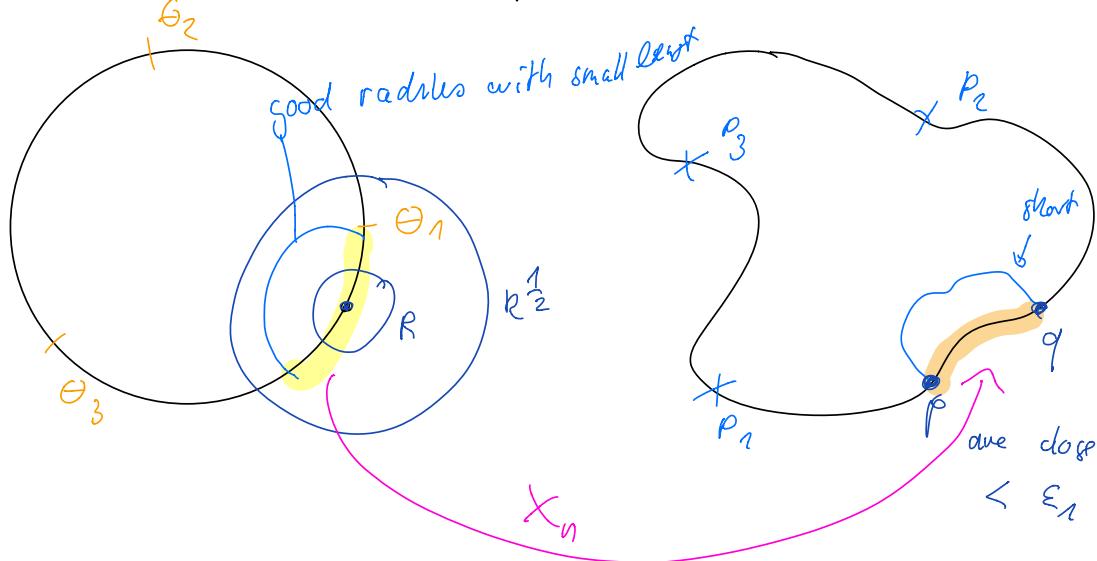


Lemma 2.4: Bounded sets of $H^*(P)$ are precompact.

Proof Idea: let $(X_n)_n \subset H^*(P)$ be bounded
i.o. $E(X_n) \leq E_0 \quad \forall n$



choose R small depending on ε_1 .



$\Rightarrow X_n \mid_{S^n}$ is equicontinuous.

\Rightarrow Apply the Direct method to find

$$E(X) = \inf_{H^*(\Gamma)} E(X) = \inf_{H^*(\Gamma)} E(X)$$

Step 3: Don't look at E alone

$$A(X) \leq E^\sigma(X) := (1-\sigma)A(X) + \sigma E(X) \leq E(X)$$

Apply (D) to $E^\sigma(X)$ on $H^*(\Gamma)$

giving a minimizer X^σ

due to Lemma 2.2 $\Rightarrow \Delta X^\sigma = 0$
 X^σ is satisfying (C)

$$\Rightarrow A(X^\sigma) = E(X^\sigma) = E^\sigma(X^\sigma)$$

Using this one shows that

$$\inf A = \inf E.$$

\Rightarrow

Theorem (The Douglas-Rado Theorem)

Let Γ be a smooth, simple closed curve in \mathbb{R}^n
and $C(\Gamma)$ as above.

$\exists X \in C(\Gamma)$ that minimizes $A(X)$

Furthermore X is a PDE solution to the
Plateau Problem.

next week.

Questions : 1) Is X a classical solution?

$$\text{if } |\partial X| \neq 0 \quad \forall z \in D$$

2) Is X unique?

3) Is X embedded?

4) Are there higher genus surfaces?

5) Does X agree with the "current"
"tails" if solution?

\cap is not real analytic.

B. White: Lectures on minimal surfaces (arXiv)

F. Struwe: Plateau's Problem and the calculus of variations

T. Colding & W. Minicozzi II: A course on minimal surfaces

III. local properties of the Douglas-Rado solution.

Let me recall:

We say $X: \underbrace{\Sigma} \rightarrow \mathbb{R}^N$ is said to be a PPE solution to Plateau's problem

Riemann surface in our case essentially $\Sigma = D$

$$\text{iff} \quad (i) \quad \Delta X = 0$$

$$A(X) = E(X) \Leftarrow (ii) \quad X \text{ almost conformal}$$

$$(iii) \quad X: \partial \Sigma \rightarrow \mathbb{P} \text{ monotonic}$$


outer
+ inner
variation

It is called a Douglas-Rado minimizer if

$$(i) \quad E(X) \leq E(Y) \quad \forall Y \in H(\Gamma)$$

$$(ii) \quad X : \partial\Sigma \rightarrow \Gamma \quad \text{monotonic}$$

last time
we showed
the existence
of a minimizer

Questions: Let $X : \Sigma \rightarrow \mathbb{R}^N$ be a Douglas-Rado minimizer.

• \Rightarrow (1) $X(\Sigma)$ an immersed surface? $\Leftrightarrow |DX| \neq 0$ on Σ

\Rightarrow (2) $X : \partial\Sigma \rightarrow \Gamma$ is a homeomorphism? $\Leftrightarrow X : \partial\Sigma \rightarrow \Gamma$ is strictly monotone

(3) $X : \partial\Sigma \rightarrow \Gamma$ an embedding?

(4) X unique?

\Rightarrow (5) $X(\Sigma)$ = to a minimizer found by the other approaches?
next week.

3.1. Branchpoints of X related to (1) & (2)

Definition 3.1: $z \in \Sigma$ is called a Branchpoint of X if $DX(z) = 0$

Lemma 3.2: ("Weierstraß parametrization")

$X : D \rightarrow \mathbb{R}^N$ satisfies (i) & (ii) $\Leftrightarrow \exists \phi : D \rightarrow \mathbb{C}^N$ holomorphic with

$$\begin{aligned} \tilde{\partial}_z := \frac{1}{2} (\partial_x - i\partial_y) &= \frac{1}{2} (\partial_1 - i\partial_2) & (a) \quad \phi^2 = 0 &\Rightarrow \langle \phi, \phi \rangle_{\mathbb{R}^N} \\ \tilde{\partial}_{\bar{z}} = \frac{1}{2} (\partial_x + i\partial_y) &= \frac{1}{2} (\partial_1 + i\partial_2) & (b) \quad X = 2 \operatorname{Re} \left(\int_0^z \phi dz \right) + X(0) \end{aligned}$$

Proof: \Rightarrow set $\phi = \tilde{\partial}_z X$ then $\tilde{\partial}_{\bar{z}} \phi = \tilde{\partial}_{\bar{z}} \tilde{\partial}_z X = \frac{1}{4} \Delta X = 0$

$$\text{to (a)} \quad \phi^2 = (\tilde{\partial}_z X)^2$$

$$= \frac{1}{4} (|\tilde{\partial}_z X|^2 - |\tilde{\partial}_{\bar{z}} X|^2) - \frac{1}{2} i \langle \tilde{\partial}_z X, \tilde{\partial}_{\bar{z}} X \rangle$$

$= 0$ since X is almost conformal.

$$\text{to (b)} \quad \text{if set } Y := 2 \operatorname{Re} \left(\int_0^z \phi dz \right) \Rightarrow Y = X + X_0$$

" we define it by (6)

$$\Delta X = \partial_{\bar{z}} \partial_z X = \partial_{\bar{z}} \phi = 0$$

$$\frac{1}{4} ((|\partial_1 X|^2 - |\partial_2 X|^2) - 2i \langle \partial_1 X, \partial_2 X \rangle) = (\phi)^2$$

□

Consequences:

(1) to find examples of minimal surfaces one can use (a) 8(b)

$$\text{In 3D: } \phi = (\phi^1, \phi^2, \phi^3) = \left(\frac{1}{2}(g^2 - g), \frac{i}{2}(g^2 + g), g \right)$$

$\begin{matrix} \text{meromorphic} \\ \text{holomorphic} \end{matrix}$

+ if z_0 is a pole of g $\Rightarrow \phi^3$ has a zero of order $2m$.

Examples:

- Catenoid $g(z) = z, \quad \phi^3 = \frac{1}{z}$
- Helicoid $g(z) = z, \quad \phi^3 = \frac{1}{z}$
- Enneper $g(z) = z, \quad \phi^3 = z$

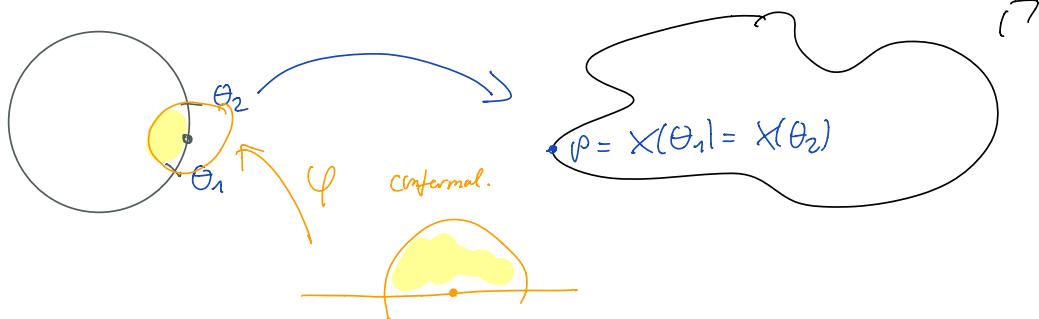
(2) The set of Branchpoints is discrete in the interior

$$\Delta X = 0 \Leftrightarrow \phi = 0$$

Theorem 3.2: Let $X: \Sigma \rightarrow \mathbb{R}^N$ be a PDE solution to Plateau's problem

$\Rightarrow X: \partial \Sigma \rightarrow \mathbb{P}$ is a homeomorphism.

Proof: Suppose not. i.e. $\exists \theta_1 < \theta_2$ s.t. $X(\theta_1) = X(\theta_2) = p$



pass to $X \circ \varphi$ and call it $X: D^+ \rightarrow \mathbb{R}^N$ and $X(x, 0) = p \quad \forall x \in [-\varepsilon, \varepsilon]$

$$\partial_2 X = \phi: D^+ \rightarrow \mathbb{C}^N \quad \text{holomorphic}$$

$$\Re(\partial_1 X - i\partial_2 X) \Rightarrow \Re(\phi(x, 0)) = 0 \quad \forall x \in [-\varepsilon, \varepsilon]$$

extend ϕ on D_ε by $\hat{\phi}(z) = \begin{cases} \phi(z) & \text{if } z \in D_\varepsilon^+ \\ \overline{-\phi(\bar{z})} & \text{if } z \in D_\varepsilon^- \end{cases}$

$\hat{\phi}$ is holomorphic.

$$\text{since } \phi^2 = 0 = \frac{1}{4}(\operatorname{Re}(\phi)^2 - \operatorname{Im}(\phi)^2) - \frac{i}{2}(\langle \operatorname{Re}(\phi), \operatorname{Im}(\phi) \rangle)$$

$$\Rightarrow \operatorname{Im}(\phi) = 0 \quad \forall x \in [-\varepsilon, \varepsilon]$$

$$\Rightarrow \phi(x) = 0 \quad \forall x \in [-\varepsilon, \varepsilon] \Rightarrow \phi = 0 \quad \square$$

Questions concerning branchpoints:

1) Can they exist? Answer: Yes
for PDE solutions

$$a) X(z) \cong (z^2, z^3) \in \mathbb{R}^4$$

b) $\ln z^3$

$$X(z) = \operatorname{Re}((z^2 - \frac{1}{4}z^4, i(z^2 + \frac{1}{4}z^4), z^3))$$

This is a
minimizer.
(next time)

2) Can they exist for minimizers? Answer: a) YES if $N \geq 4$

\Rightarrow b) No if $N = 3$
task for today

Corollary 3.2 (Local structure I)

$$\text{if } 0 \in \Sigma \text{ then } X(z) = (\underbrace{z^{m+1}}_{\in \mathbb{R}^2 \cong \mathbb{C}}, 0) + k(z) \text{ with } k(z) \in O(|z|^{m+2})$$

and $m \in \mathbb{N} \cup \{0\}$
(it is a branchpoint if $m \neq 0$)

$$\text{and } \Theta(X(\Sigma), 0) = (m+1)$$

Proof: ϕ holomorphic

$$\phi(z) = (m+1) \cdot a \cdot z^m + b z^{m+1} + \dots$$

$$0 = \phi^2(z) = (m+n)^2 \alpha^2 z^{2m} + b \cdot \alpha z^{2m+1} + \dots$$

$$\Rightarrow \alpha^2 = 0 \quad \text{or} \quad b \cdot \alpha = 0$$

$$\alpha \in \mathbb{C}^N \Rightarrow \alpha = \alpha_1 + i\alpha_2 \quad \text{with } \alpha_1, \alpha_2 \in \mathbb{R}^N$$

$$\alpha^2 = 0 \Leftrightarrow 0 \neq |\alpha_1| = |\alpha_2| \quad \& \quad \alpha_1 \perp \alpha_2$$

actually after a rotation (* scaling of the parametrization $\frac{1}{z}\alpha_1 = e_1, \frac{1}{z}\alpha_2 = e_2$)

Remark: $\mathcal{Q} = \{\alpha \in \mathbb{C}^N : \alpha^2\} \cong G(2, n)$ the two planes in \mathbb{R}^N

$$\int_0^z \phi dz = \underbrace{\alpha z^{m+1}}_{\frac{1}{2}(e_1 + ie_2)} + b(z) \quad \text{with} \quad b(z) = \mathcal{O}(|z|^{m+2})$$

$$\Rightarrow X(z) = (\underbrace{\operatorname{Re}(z^{m+1}), \operatorname{Im}(z^{m+1}), 0}_0) + \underbrace{\frac{2\operatorname{Re}(b(z))}{\mathcal{O}(|z|^{m+2})}}$$

by identifying \mathbb{P}^2 with \mathbb{C} by

$$X^1 - iX^2 = z^{m+1} + \mathcal{O}(|z|^{m+1})$$

$$\text{One may use that.} \quad 2|\partial_z X|^2 = \frac{1}{2}|DX|^2 = 2|\phi|^2$$

$$\geq 2(m+1)^2 |z|^{2m}$$

$$\text{to calculate} \quad \Theta(X, 0) = (m+1)$$

Corollary 3.3 if $X: \Sigma \rightarrow \mathbb{R}^N$ is a PDE solution

$$\text{with } P = X(\partial\Sigma) \quad \text{and} \quad \int_P |\kappa_P| < \epsilon_\Omega$$

$\Rightarrow X$ is embedded + P without branch points.

Proof: See lecture 1. □

Lemma 3.4. (local structure II.)

Let $z_0 = 0$ be a branch point of X

then $\exists \psi: U \ni 0 \rightarrow V \ni 0$ local diffeomorphism (analytic)
and $m > 0$

$$\text{s.t. } (X \circ \psi)(z) = (z^{m+1}, h(z)) \quad \text{where } h(z) = O(|z|^{m+2})$$

$\overset{\text{D}^2}{\mathbb{D}^2}$ $\overset{\text{D}^{N-2}}{\mathbb{R}^{N-2}}$

Proof: by Corollary 3.3. we have.

$$\begin{aligned} \mathbb{C} \ni (X^1 - iX^2)(z) &= z^{m+1} + \underbrace{h(z)}_{\text{not holomorphic}} \quad \text{with } h(z) = O(|z|^{m+2}) \\ &= \left(z \left(1 + \frac{h(z)}{z^{m+1}} \right)^{\frac{1}{m+1}} \right)^{m+1} \end{aligned}$$

now set $\varphi(z) = z \left(1 + \frac{h(z)}{z^{m+1}} \right)^{\frac{1}{m+1}}$

φ is analytic around 0. $(1 \leq \frac{1}{2} \text{ if } |z| \ll 1)$

$$\partial_z \varphi(z) = 1 + O(z)$$

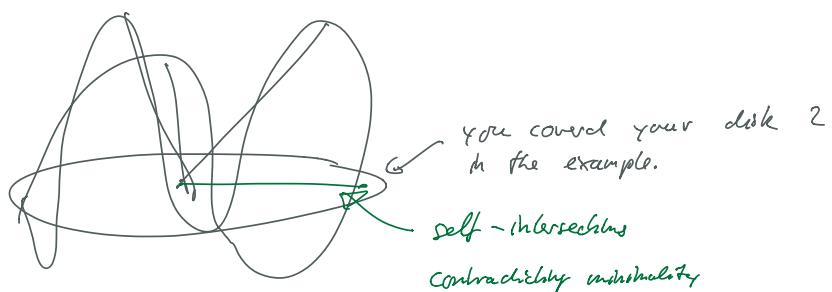
$$\partial_{\bar{z}} \varphi(z) = O(z)$$

so $\varphi = \varphi^1$ exists around 0 and is real analytic.

$$\Rightarrow (X \circ \varphi) = (z^{m+1}, h(z))$$

□

Recall:



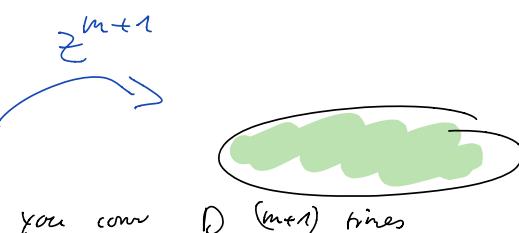
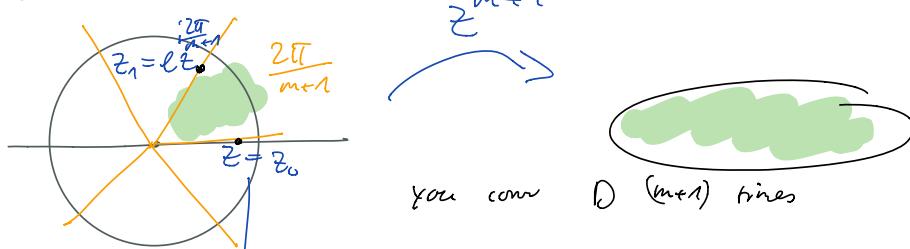
How to find self-intersecting lakes:

Definition 3.5: Let $X(z) = (z^{m+1}, h(z))$ be the representation of

Lemma 3.3 and set $d(z) := h(e^{i\frac{2\pi}{m+1}}z) - h(z)$
 then. $0 \neq a$ $\begin{cases} \text{TRUE Branchpt if } h \neq 0 \\ \text{False Branchpt if } h = 0 \end{cases}$

Since if $h(z) = 0$ then $X(z) = (z^{m+1}, \tilde{h}(z^{m+1}))$
 for some \tilde{h}

Geometric idea behind d :



What happens with h if you turn by $\frac{2\pi}{m+1}$ is measured by d :
 $d(z) = h(z_1) - h(z_0)$

so if $d \leq 0$ then $h(e^{i\frac{2\pi}{m+1}}z) = h(z) \quad \forall l=1, \dots, m+1$

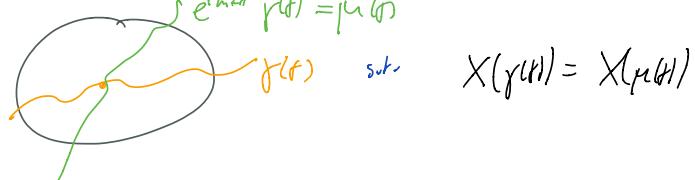
so we cover the same image $(m+1)$ -times

Idea: $z \mapsto (z^2, 0)$ is a double cover of D .

Idea: How to rule out true branchpts for minimizers
 in \mathbb{R}^3 .

1. Step: $d: D \rightarrow \mathbb{R}$ has 0 as isolated branchpt

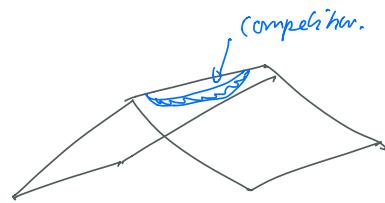
and $\exists \gamma(t)$ in D s.t. $\gamma(0) = 0$ and $d(\gamma(t)) = 0$



2. step: You show that

is not minimally.

by smoothing out the corners.



3. step: Show that if γ & μ exists.

then $\exists \phi: D \rightarrow D$ which is locally Lipschitz
(not continuous along a line) s.t.

$$(a) X \circ \phi = X$$

$$(b) X \circ \phi \text{ contains } \begin{array}{c} \text{a} \\ \diagup \\ \text{triangle} \\ \diagdown \end{array}$$

Combining (1) - (3) $\Rightarrow X$ cannot be minimally.

to Step 2: You show that $h(\cdot)$ and $h(e^{i\frac{\pi}{m}} \cdot)$
are solving the same PDE.

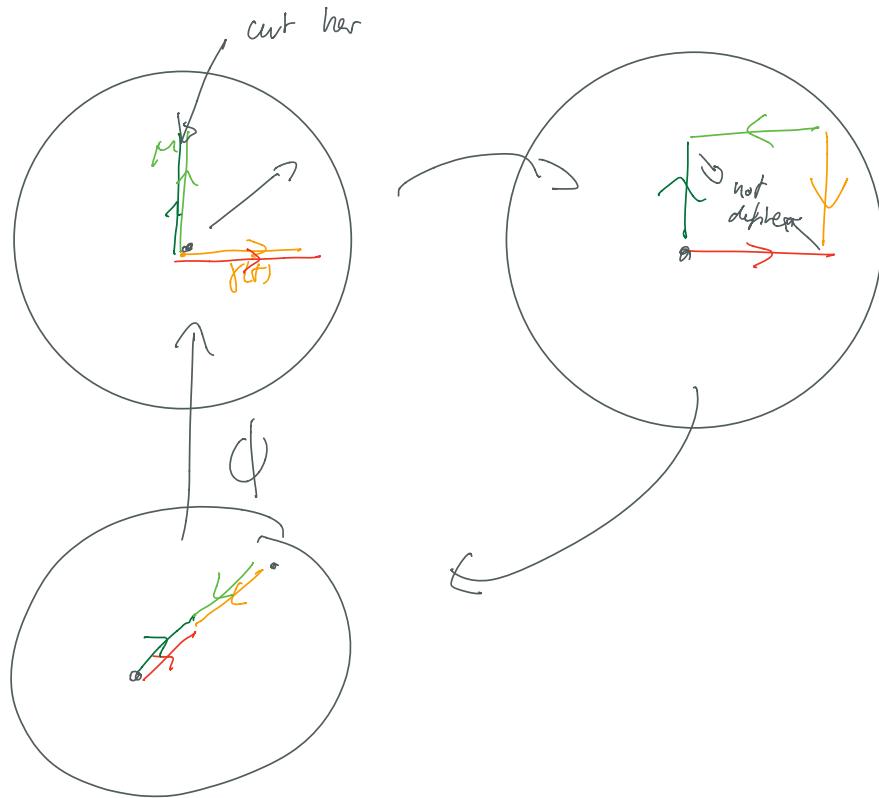
$$\text{since } \Delta_g X^3 = 0 \Leftrightarrow \Delta_g h = 0 \quad \Delta_g h|_1 = 0$$

$$\text{and so you } \Delta_g d \stackrel{?}{=} 0 \Leftrightarrow \text{div}(A(z) \nabla d|_1 = 0 \quad A(0) = \text{id}.$$

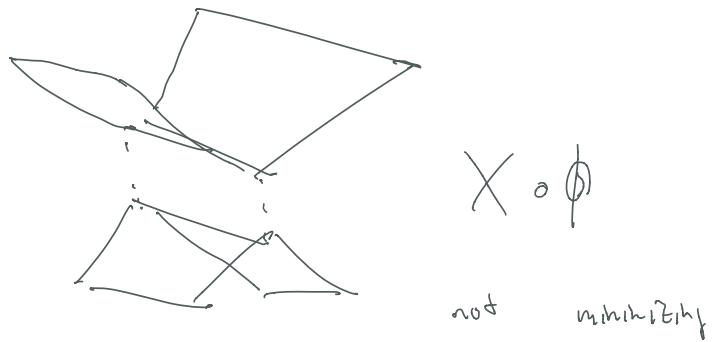
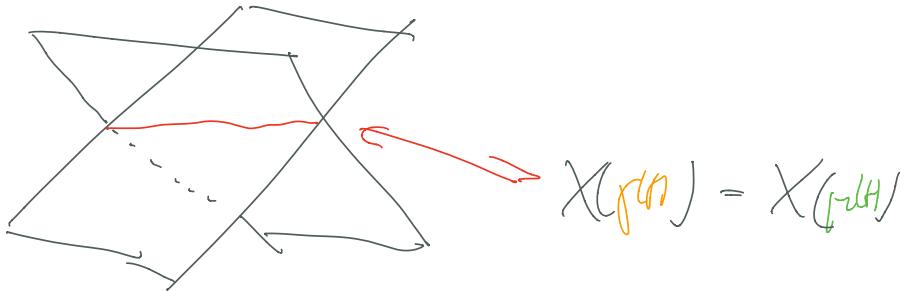
$$d(z) = \operatorname{Re}(z^\ell) + O(z^{\ell+1})$$

to Step 2: clear:

to step 3:



locally

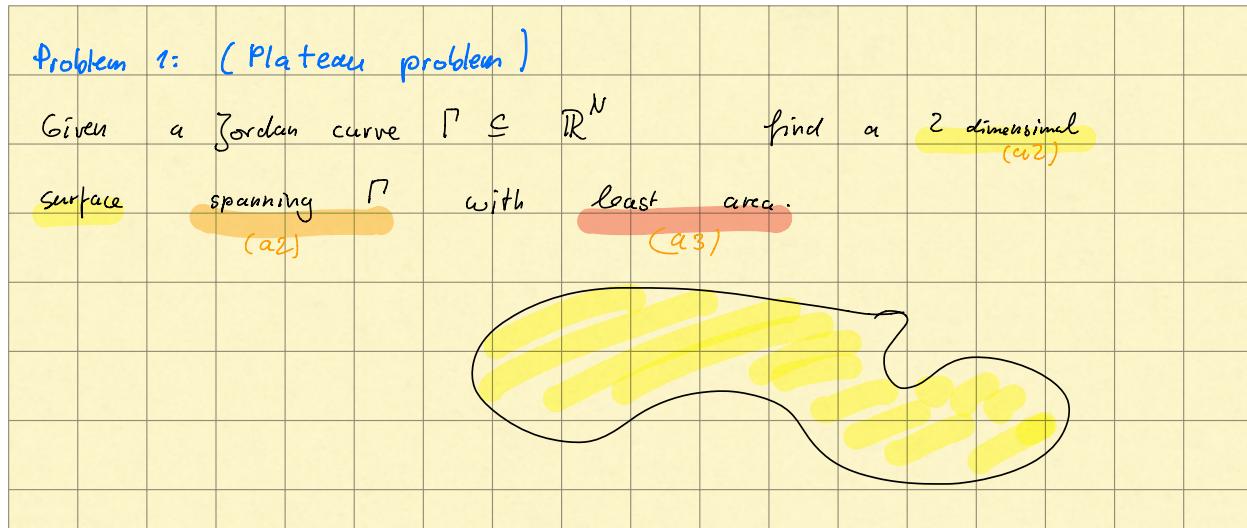


The Question is if TRUE branchpoint can occur on Γ
in \mathbb{R}^3 is open if Γ is not real analytic.

Remark: if Γ is Jordan curve then there
are no FALSE branchpts.

Ringvorlesung 4.

(Non classical minimizing surfaces with smooth boundary
j.w. C. De Lellis, G. De Philippis)



- Q:**
- Find a good class \mathcal{M} of "surfaces" to minimize
 - Concerning the minimizer \Rightarrow its properties.
 - Uniqueness (\mathcal{M} is not convex)
 - Regularity (Interior/boundary - smoothness, branchpoints...)
 - Topology (what genus ...)

\Rightarrow Do the answers depend on \mathcal{M} .

We considered so far the Douglas-Rado approach $(^{\text{139}})$:

It's parametric.

Idea: Fix an abstract Riemannian Σ_g of genus g (for us it was $\Sigma_g = \mathbb{D}$) s.t. $\partial\Sigma_g$ has a single component.

$$\mathcal{M}_g(\Gamma) := \left\{ X: \Sigma_g \rightarrow \mathbb{R}^N \text{ locally Lipschitz : } X: \partial\Sigma_g \rightarrow \Gamma \right. \\ \left. \text{monoton parametrisation (of degree 1)} \right\}$$

$$m_g^D(\Gamma) := \min_{\sum g} \left\{ \frac{1}{2} \int |DX|_h^2 d\text{vol}_h : X \in M_g(\Gamma), h \text{ a metric on } \Sigma \right\}$$

only necessary for $g < \infty$

for h fixed it works with our approach \Rightarrow PDE theory

the variation in $h \Rightarrow$ Teichmüller theory \Rightarrow the minimizer will be conformal.

Federer - Fleming ('60): (building on work by Giorgi '55)

measure theoretic generalization of smooth oriented surfaces \Rightarrow integral current.

let Σ be a smooth 2d-surface and $\omega \in \mathcal{D}(\mathbb{R}^N; \Lambda^2(\mathbb{R}^N))$

$$\underline{[\Sigma]}(\omega) = \int_{\Sigma} \omega \xrightarrow{\text{generalize it}} \int_{\Sigma} \Theta \omega \quad \text{with } \Theta: \Sigma \rightarrow \mathbb{Z}$$

measurable.

$$\Rightarrow \underline{[\Sigma]}(\omega) = \int_{\Sigma} \omega \quad \begin{matrix} \uparrow \\ \text{2-form} \end{matrix}$$

\Rightarrow let $\{\Sigma_i\}_i$ be a family of 2d- C^1 -surfaces with $\Sigma_i \cap \Sigma_j$
and $\Theta_i: \Sigma_i \rightarrow \mathbb{Z}$ measurable.

$$\Rightarrow T(\omega) := \sum_i \int_{\Sigma_i} \Theta_i \omega \quad (\star)$$

$$\Rightarrow \text{mass of the current } T: \quad \lVert T \rVert = \sum_i \int_{\Sigma_i} |\Theta_i|$$

$$\lVert T \rVert([\Sigma]) = \sum_i 1 \text{ vol}_h = \text{Area}(\Sigma)$$

Note: if $\phi: \Sigma \rightarrow \mathbb{R}^N \in M_g(\Gamma)$

$$(\phi_* [\Sigma])(\omega) = \int_{\Sigma} \phi^* \omega \quad \Rightarrow \quad M_g(\Gamma) \subseteq I^2(\mathbb{R}^2)$$

↑ the "closure"
of currents generated by (\star)

$$\Rightarrow m^F(\Gamma) := \min \{ M(T) : T \in \mathcal{I}^2(\mathbb{R}^2) \text{ & } \partial T = \Gamma \}$$

Observation: $m^F(\Gamma) \leq m_g^D(\Gamma)$ $\forall g$

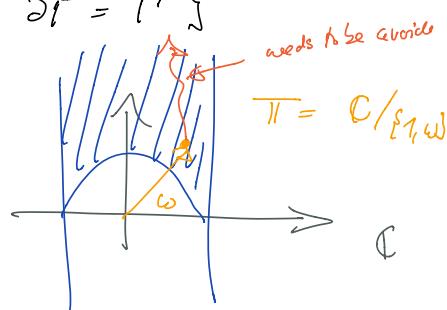
To Question 1: Existence:

$$\mathcal{J}(\Gamma) \cong$$

For Douglas - Rado:

We had seen existence for $g=0$: the disk.

You have existence if $m_g^D(\Gamma) < m_{g-1}^D(\Gamma)$ [Douglas, Schiffermann, Courant, Jost]



For Feder - Fleming:

"Banach - Alaoğlu" because $\mathcal{I}^2(\mathbb{R}^N) \subseteq \mathcal{D}^*(\mathbb{R}^N; L^2(\mathbb{R}^N))$

+ closure theorem of Feder - Fleming:

$\{ T \in \mathcal{I}^2(\mathbb{R}^N) : \Pi(T) + \Pi(\partial T) \leq 1 \}$ are weakly-closed.

To Question 2: Regularity

for Douglas - Rado:

Answser: $\underline{\phi}$ will be smooth by PDE techniques $\Delta_n \phi = 0$

away from branchpoints if it's a classical immersion:

- Branchpoints: interior:
- at most countable
 - ruled out for $N=3$
 - can appear for $N \geq 3$

At the boundary: only known for Γ real analytic then they don't occur. (B. White)

for the Feder - Fleming:

- in the interior:
- $N=3 \Rightarrow$ smooth embedding (di Giorgi)
 - $N > 3$ (Almgren, Cheeger, De Lellis-Spadaro-Spoliarz)
singularities is a discrete set.

- at the boundary
- $N=3 \Rightarrow$ smooth (Hausdorff-Sullivan)
 - $N > 3 \Rightarrow$ regular set is open-closed.

\Rightarrow Question: $m_g^D(\Gamma) = m^F(\Gamma)$ for some g ?

a minimizer of Federer-Fleming

If YES then g must be finite $\Leftrightarrow g(\Gamma) < \infty$

For:

$N=3$

As a conclusion of boundary + interior regularity for a Federer-Fleming minimizer

$\Gamma \Rightarrow S(\Gamma) < \infty \Rightarrow m_g^D(\Gamma) = m_{g-1}^D(\Gamma) \quad \forall g$

(one needs $\text{length}(\Gamma) < \infty$ & Γ is C^2
it's not true $\text{length}(\Gamma) = +\infty$ (Fleming))

$N > 3$: B. White: if Γ real analytic & genus is finite
 \Rightarrow no branch points at the boundary.

Conjecture: holds true with genus assumption.

Thm: For every $\varepsilon > 0, N > 0$ $\exists g$ smooth metric on \mathbb{R}^4 , smooth Jordan curve $o \in \Gamma \subseteq B_1$ of finite length, $\Sigma \subset B_1$ smooth on $B_1 \setminus \{o\}$ with

(i) $\|g - S_{\mathbb{R}^4}\| < \varepsilon$, $g = S_{\mathbb{R}^4}$ on B_1^C

(ii) there is an almost Kähler structure $\underline{\mathcal{J}}$ on \mathbb{R}^4 with almost Kählerform ω

(iii) $\partial \overline{\partial} \underline{\mathcal{J}} = [\Gamma]$; $\underline{\mathcal{J}}^\# \omega = \text{vol}_\Sigma$

(iv) Σ has infinite topology

(iv)* Σ has branch singularities accumulating at o .

$$\text{(i)} \quad \underline{\mathcal{J}}^2 = -\text{id}$$

$$\text{(ii)} \quad g(\underline{\mathcal{J}}v, \underline{\mathcal{J}}w) = g(v, w)$$

$$\text{(iii)} \quad \omega(v, w) = -g(v, \underline{\mathcal{J}}w)$$

$\rightarrow \Sigma$ is the unique minimizing current spanning Γ is a closed form
(within $m(\Gamma) = \mathcal{D}^2(\mathbb{R}^4) \cap \{\partial T = \Gamma\}$)

Idea of the construction:

- take a counterexample to boundary regularity (slightly inspired by Gulliver - an atypically branch point at boundary)
- uses heavily the almost Kähler structure.

\Rightarrow the Calibration argument:

A 2-form ω calibrates Σ if

$$i) \quad d\omega = 0$$

$$(ii) \quad \| \omega \|_* \leq 1 \quad (\text{i.e. } |\omega(v, w)| \leq \|v\| \|w\| \quad \forall v, w)$$

$$(iii) \quad i_S^* \omega = \text{vol}_S \quad (\text{i.e. if } \tau_1, \tau_2 \text{ are OND for } T_p \Sigma \\ \Rightarrow \omega_p(\tau_1, \tau_2) = 1)$$

\Rightarrow (i) Σ is a minimal surface (it's a minimizer in $\mathcal{I}^2(\mathbb{R}^N)$)

(ii) if P is $C^{1,1}$ and ω is Kähler then Σ is unique

Proof: Let $T \in \mathcal{I}^2(\mathbb{R}^N)$ with $\partial T = \partial[\Sigma]$.

Since \mathbb{R}^N is contractible $\exists \alpha \times 1\text{-form s.t. } \omega \stackrel{(ii)}{=} d\alpha$

$$\begin{aligned} \Rightarrow \text{Area}(\Sigma) - \text{Area}(T) &\stackrel{(ii)}{=} [\Sigma](\omega) - [T] \leq [\Sigma](\omega) - [T(\omega)] \\ &= [\Sigma](d\alpha) - T(d\alpha) = \partial[\Sigma](\alpha) - \partial T(\alpha) = 0 \end{aligned}$$

if " \leq " $\Rightarrow \omega$ calibrates $T \Rightarrow i_T^* \omega = \text{vol}_T$

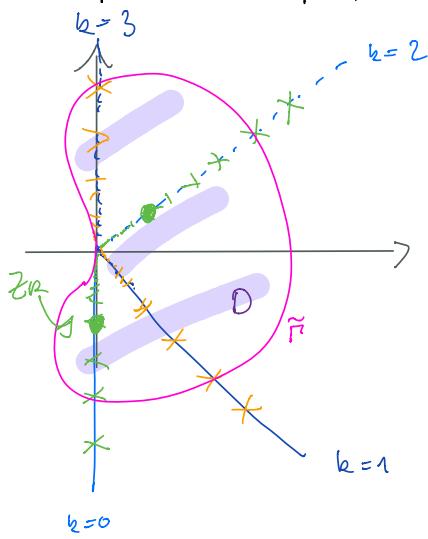
$$\Leftrightarrow T_p T = \{v, \bar{J}v\}$$

due to boundary regularity \exists regular point p on P

$$\Rightarrow T_p T = T_p \Sigma = \left\{ \frac{\tau}{p}, \bar{J}\tau \right\} \stackrel{\text{PDE}}{\Rightarrow} \Sigma = T .$$

tangent to P Lebesgue calibration principle

Super Sketchy proof:



$$f_k(z) = e^{-z^k} \sin(\operatorname{Im}(z) + i(\frac{\pi}{2} - k\frac{\pi}{3}))$$

with $k = 0, 1, 2, 3$

is holomorphic on $\{x > 0\}$ with smooth extension
to $\{x \geq 0\}$

$$\text{It has zeros in } \tilde{\Sigma}_k = \left\{ e^{k\pi + i(k\frac{\pi}{3} - \frac{\pi}{2})} : k \in \mathbb{Z} \right\}$$

$$\text{So } f(z) = \prod_{k=0}^3 f_k(z)$$

$$G(z) = (z^3, g(z))$$

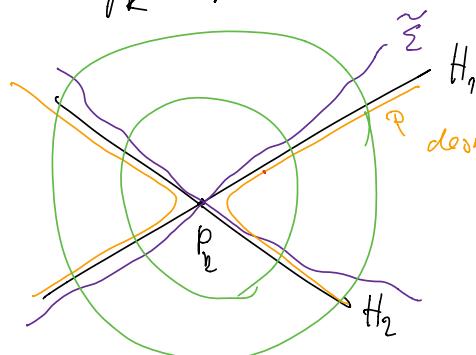
$\tilde{\Sigma}$ is holomorphic

$$P = G(\tilde{P}) \quad \tilde{\Sigma} = G(D)$$

$\tilde{\Sigma} \triangleq$ calibrated by $\omega_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$

$$G(z_k) = G(e^{i\frac{2\pi}{3}} z_k) =: p_k$$

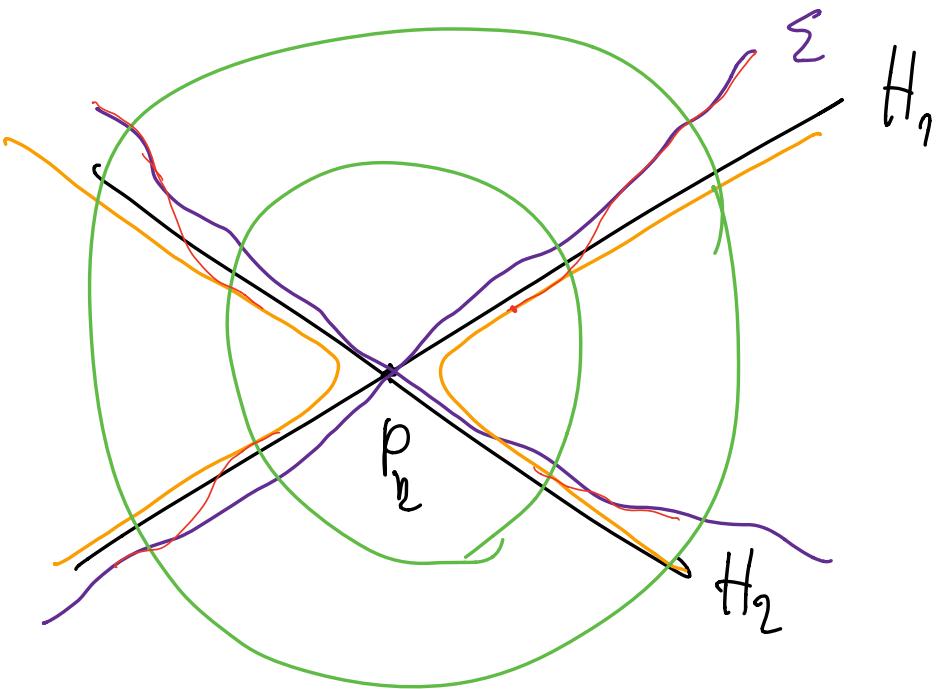
in p_k you have transversal intersection.



$$H_1 = p_1 + \{ a_1 z_1 + a_2 z_2 = 0 \}$$

$$H_2 = p_2 + \{ b_1 z_1 + b_2 z_2 = 0 \}$$

$$\text{for } a \neq b \in \mathbb{CP}^1$$



1. step: passing from Σ to Σ

2. step: modify ω_0 to find ω closed s.t.

$$\iota_{\Sigma}^*\omega_0 = \iota_{\Sigma}^*\omega \quad (\text{v. Poincaré lemma})$$

3. step: use ω as an "almost" Kähler form to find the metric g .

\square

A nice exercise: let $G = \{(x, u(x)) : x \in \mathbb{R}, u \in C^2(\mathbb{R})\}$ a minimal graph i.e. $-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$

$$\begin{aligned} \omega_{(x, u(x))} &= N \perp (dx^1 \wedge dx^2 \wedge \dots \wedge dx^{N+1}) \\ &= \det(N, \dots, 1) \\ \text{where } N(x) &= \frac{1}{\sqrt{1+|u'|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix} \end{aligned}$$

ω calibrates G in $S \times \mathbb{R}$

if \mathcal{Q} is convex then G is the unique minimizer in
 $\mathbb{I}^N(\mathbb{R}^{M \times N}) \cap \{\partial T = \partial G\}$