Representation Theory

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Symmetry is the key to many applications and computations in algebra. Symmetry is expressed via the action of a group acting on a space. Some of the most important groups are

- $\mathrm{GL}(V) = \mathrm{GL}(\dim V)$ - the group of linear isomorphisms of a finite-dimensional vector space V, with the structure of the algebraic variety given by Exercise 8 in Lecture 2;
- $\mathrm{SL}(V) = \mathrm{SL}(\dim V)$ - the group of volume and orientation preserving linear automorphisms of V, with the structure of an algebraic variety given by the equation $\det A = 1$;
- S_n - the group of permutations of a set with n elements; this is an algebraic variety consisting of $n!$ distinct points in $\mathrm{GL}(n)$, namely the $n \times n$ permutation matrices.

The groups that we consider have two structures: of an abstract group and of an algebraic variety. We note that basic group operations, like inverse or group action, are in fact morphisms of algebraic varieties. We call such groups *algebraic*. In our lecture we restrict our attention to algebraic groups and morphisms between them that are both group morphisms and morphisms of algebraic varieties. We work over an algebraically closed field K of characteristic zero.

In general, the following strategy to study an object can be very powerful:

- consider all maps from (resp. to) this object into (resp. from) another basic object.

This very general approach could be seen as motivation to study homotopy, homology or the theory of embeddings. For groups, we obtain the following central definition.

Definition 1. A representation *of a group G is a morphism* $G \to \mathrm{GL}(V)$.

Given a representation $\rho : G \to \mathrm{GL}(V)$, every element of g induces a linear map $\rho(g) : V \to V$. It is useful to think about a representation as a map $G \times V \to V$ with the notation

$$gv \ := \ \rho(g)(v) \in V.$$

Here, we have the natural compatibilities

$$(g_1 g_2)v = g_1(g_2 v) \quad \text{and} \quad g(\lambda v_1 + v_2) = \lambda g v_1 + g v_2,$$

where $\lambda \in K$, $v, v_1, v_2 \in V$ and $g, g_1, g_2 \in G$. We say that the group G *acts on* the vector space V. If the action follows from the context then we call V a *representation* of G.

Example 2. The groups GL(n) and SL(n) act (by linear change of coordinates) on the space $V = K[x_1, \ldots, x_n]_k \simeq K^{\binom{n+k-1}{k}}$ of homogeneous polynomials of degree k in n variables. Using the monomial basis on V, the representation ρ maps a small matrix, of size $n \times n$, to a large matrix, with rows and columns indexed by monomials of degree k. The entries in that large matrix are homogeneous polynomials of degree k in the entries of the small matrix. We recommend working this out for $n = k = 2$. This representation ρ of GL(n) plays an important role in classical *Invariant Theory*, the topic to be studied in the next lecture.

The representations of a fixed group G are the objects of a category. In this category, a morphism f between representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ is a linear map $f : V_1 \to V_2$ that is compatible with the group action:

$$f(\rho_1(g)(v)) = \rho_2(g)(f(v)) \quad \text{for all } g \in G \text{ and } v \in V_1.$$

This can also be written as $f(gv) = gf(v)$. The category of representations of a group G is an *abelian category.* This means in particular that kernels and cokernels exist - cf. Exercise 2.

Our first aim is to describe the basic building blocks of representations.

Definition 3. *A* subrepresentation *of a representation V of a group G is a linear subspace $W \subset V$ such that the action of G restricts to W, i.e.*

$$gw \in W \text{ for all } w \in W \text{ and } g \in G.$$

Equivalently, a subrepresentation is an injective map in the category of representations.

Note that for any representation V, the subspaces 0 and V are always subrepresentations.

Definition 4. *A representation V is called* irreducible *if and only if 0 and V are its only subrepresentations. We next show that there are no nonzero morphisms between nonisomorphic irreducible representations.*

Lemma 5 (Schur's Lemma)**.** *Let V_1 and V_2 be irreducible representations of a group G. If $f : V_1 \to V_2$ is a morphism of representations then either f is an isomorphism or $f = 0$. Further, any two isomorphisms between V_1 and V_2 differ by a scalar multiple.*

Proof. Both the kernel $\ker f$ and the image $\operatorname{im} f$ are representations. As V_1 is irreducible, either $\ker f = V_1$ or f is injective. In the latter case, $\operatorname{im} f \simeq V_1$ is a nontrivial subrepresentation of V_2, hence f is also surjective, i.e. it is a linear isomorphism. The inverse of f, as a linear map, is also the inverse as morphism of representations.

For the last part, consider two isomorphisms f_1 and f_2. We may assume that f_1 is the identity on V_1. Let v be the eigenvector of f_2 with eigenvalue $\lambda \in K$. We have:

$$f_2(v) = \lambda v = \lambda f_1(v).$$

Consider the morphism of representations $f := f_2 - \lambda f_1$. Clearly, $v \in \ker f$. Hence, by the first part, $f_2 - \lambda f_1$ is the zero map, and hence $f_2 = \lambda f_1$. □

Theorem 6 (Maschke's theorem)**.** *Let V be a finite-dimensional representation of a finite group G. There exists a direct sum decomposition*

$$V \;=\; \bigoplus V_i,$$

where each V_i is an irreducible representation of G.

Proof. By induction on the dimension, it is enough to prove the following statement: if W is a subrepresentation of V, then there exists a subrepresentation W' such that $V = W \oplus W'$.

Let $\pi : V \to W$ be any (surjective) projection. Let $\tilde{\pi} : V \to W$ be defined by:

$$\tilde{\pi}(v) \;=\; \frac{1}{|G|}\sum_{g\in G} \rho(g)_{|W} \circ \pi \circ \rho(g)^{-1}.$$

We note that $\tilde{\pi}$ is a morphism of representations and $V = W \oplus \ker\tilde{\pi}$. □

Remark 7. *The existence of decomposition into irreducible components holds not only for finite groups. It also holds for* $\mathrm{GL}(n)$ *and* $\mathrm{SL}(n)$*. One possible proof is similar to the one above and is known as the* unitarian trick*. It was introduced by Hurwitz and generalized by Weyl. A representation that allows such a decomposition is called* semi-simple *or* completely reducible*. If all representations of G have this property then the group G is called* reductive*.*

The decomposition of into irreducible representations in Maschke's Theorem is not unique. The following example makes this clear.

Example 8. Any group G acts on any vector space V trivially by $gv = v$. Any subspace of V is a subrepresentation. The irreducible subrepresentations are the 1-dimensional subspaces of V. Hence, any decomposition into 1-dimensional subspaces $V = \bigoplus_{i=1}^{\dim V} K^1$ is a decomposition into irreducible representations, but there is no distinguished one.

As we will see, the reason for nonuniquness, is the fact that distinct V_i's appearing in the decomposition may be isomorphic. Let us group the isomorphic V_i's together obtaining:

$$V \;=\; \bigoplus V_j^{\times a_j}, \tag{1}$$

where $V_{j_0} \simeq V_{j_1}$ if and only if $j_0 = j_1$. The subrepresentations $V_j^{\times a_j}$ are called *isotypic components*. The number a_j is the *multiplicity* of the irreducible representation V_j in V.

Corollary 9 (to Schur's Lemma)**.** *The isotypic components and multiplicities of a semi-simple representation V are well defined, i.e. do not depend on the choice of the decomposition into irreducible representations.*

Proof. Consider two decompositions:

$$V = \bigoplus_j V_j^{\times a_j} = \bigoplus_k V_k^{\times b_k}.$$

Allowing a_j, b_k to be equal to zero, we may assume that all irreducible representations occur and that the indexing in both sums $\bigoplus$ is the same. First we prove that for a given irreducible representation V_i we have $a_i = b_i$. The restriction of identity gives us an injective map:

$$m : V_i^{\times a_i} \to \bigoplus_k V_k^{\times b_k}.$$

By Schur's Lemma, the composition of m with the projection

$$\pi_s : \bigoplus_k V_k^{\times b_k} \to V_s^{\times b_s}$$

equals zero, unless $s = i$. Hence, $\operatorname{im} m \subset V_i^{\times b_i}$. In particular, by dimension count, $a_i \leq b_i$. Analogously $b_i \leq a_i$, i.e. the multiplicities do not depend on the decomposition. Further, the composition $\pi_s \circ m$ is an isomorphism if $s = i$ and is zero if $s \neq i$. It follows that $\operatorname{im} m = V_i^{\times b_i}$. Thus, the identity maps isotypic components to (the same) isotypic components. □

Our next aim is to understand the irreducible representations of a given group G. The following definition provides us with the most important tool.

Definition 10 (Character)**.** *Let $\rho : G \to GL(V)$ be a representation of G. The* character *$\chi_\rho = \chi_V$ of ρ is the function $G \to K$ obtained by composing ρ with the trace function* Tr*:*

$$\chi_\rho(g) \;=\; \operatorname{Tr}(\rho(g)).$$

The properties of the trace of a square matrix imply the following facts about characters:

- If $V = \bigoplus V_i$ then $\chi_V = \sum \chi_{V_i}$.
- If g_1 and g_2 are conjugate elements of G, then $\chi(g_1) = \chi(g_2)$ for any character χ.
- If V_1, V_2 are representations with characters χ_1, χ_2 then their tensor product $V_1 \otimes V_2$ is also a representation, and its character is the product $\chi_1\chi_2$.
- We have $\chi_V(e) = \dim V$, where $e \in G$ is the neutral element.

For a finite group G, we fix the following scalar product on the space of functions $G \to \mathbb{C}$:

$$\langle \chi_1, \chi_2 \rangle \;:=\; \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}. \tag{2}$$

It turns out that characters of all irreducible representations of G are orthonormal with respect to this scalar product. For details we refer to Serre's book [2, Chapter 2]. In particular, the characters are linearly independent elements in $\mathbb{C}^G$. Hence, we can find the multiplicities a_j in the isotypic decomposition $V = \bigoplus_j V_j^{a_j}$ by decomposing the character:

$$\chi_V \;=\; \sum_j a_j \chi_j.$$

For any finite group G there are finitely many irreducible representations - the sum of squares of their dimensions equals the order of the group [2, Chapter 2.5, Corollary 2]. A *class function* if a function $G \to K$ that is constant on conjugacy classes. Characters in fact form a basis of the space of class functions. Often (all) characters are represented in a table, which makes the decomposition very easy, if we know the character of a representation.

Example 11. We present the character table for the symmetric group S_3 on three letters:

	Trivial representation	Sign representation	2-dimensional repr.
1 identity	1	1	2
2 cycles (ijk)	1	1	-1
3 transpositions (ij)	1	-1	0

There are three conjugacy classes, hence there are three irreducible representations. The first is the trivial representation $gv = v$, the second is the sign representation $gv = (\operatorname{sgn} g)v$, and the third is the two-dimensional representation, given by the symmetries of a regular triangle. Each column in the table represents a function $S_3 \to \mathbb{C}$. Make sure to check these functions are orthonormal with respect to the inner product (2). In fact, one builds the character table of a finite group by exploiting the orthonormality of the columns. In this manner, one obtains the 5×5 character table for S_4 and the 7×7 character table for S_5.

These ideas generalize to $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. However, we cannot represent their characters by tables. However, we can represent each character χ by its values on the Zariski dense subset of diagonalizable matrices. Hence, we fix a torus $T = (K^*)^n \subset \mathrm{GL}(n)$ and restrict the character to T. As χ is constant on conjugacy class and any diagonalizable matrix is conjugate to an element of T, the function $\chi_{|T}$ characterizes χ. Therefore, given any representation W of $\mathrm{GL}(n)$, we restrict the group and regard W as a representation of T. By Exercise 1 and Corollary 9 we know that, as a representation of T, the space W decomposes:

$$W = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} W_{\mathbf{b}}^{a_{\mathbf{b}}}, \tag{3}$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ takes w to $\mathbf{t}^{\mathbf{b}} w$ for $w \in W_{\mathbf{b}}$. The isotypic components $W_{\mathbf{b}}^{a_{\mathbf{b}}}$ for the T-action are called *weight spaces*. The characters $\mathbf{b}$ of T for which $a_{\mathbf{b}} \neq 0$ are called *weights*.

Remark 12. Let T be the torus of diagonal matrices $\mathbf{t} = \operatorname{diag}(t_1, \ldots, t_n)$ in $\mathrm{GL}(n)$. If χ is a character of $\mathrm{GL}(n)$ then its restriction to T is the function $\chi|_T : T \to K$, $\mathbf{t} \mapsto \operatorname{Tr}(\rho(\mathbf{t}))$. Here Tr denotes the trace of a (large) square matrix. The restricted character $\chi|_T$ equals

$$\chi|_T(\mathbf{t}) = \sum_{\mathbf{b} \in \mathbb{Z}^n} a_{\mathbf{b}} \mathbf{t}^{\mathbf{b}}.$$

This is a Laurent polynomial in $t_1, \ldots, t_n$ that is invariant under permuting these n unknowns

Example 13. Following Example 2, we consider the action of $\mathrm{GL}(n)$ on homogeneous polynomials of degree k. Let χ be its character. Then $\chi|_T$ is the *complete symmetric polynomial* of degree k, i.e., $\chi|_T(\mathbf{t})$ is the sum of all monomials $\mathbf{t}^{\mathbf{a}}$ where $\mathbf{a} \in \mathbb{N}^n$ and $|\mathbf{a}| = k$.

Example 14. The group $\mathrm{GL}(n)$ acts naturally on the kth exterior power $V = \wedge_k K^n$. Write ρ for this representation and χ for its character. We identify V with $K^{\binom{n}{k}}$ by fixing the standard basis $\left\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\right\}$. The image $\rho(g)$ of an $n \times n$-matrix $g = (g_{ij})$ is the *kth compound matrix* or *kth exterior power*, whose entries are the (suitably signed) $k \times k$ minors of g. We note that the determinant of $\rho(g)$ equals $\det(g)^{\binom{n-1}{k-1}}$. The restricted character $\chi|_T(\mathbf{t})$ is the kth elementary symmetric polynomial in $t_1, \ldots, t_n$.

For a concrete example, let $k = 2$. Then $\rho(g)$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix whose rows and columns are labeled by ordered pairs from $\{1, 2, \ldots, n\}$, and whose entry in row $(i < j)$ and column $(k < l)$ equals $g_{ik}g_{jl} - g_{il}g_{jk}$. We have $\det(\rho(g)) = \det(g)^{n-1}$ and $\chi|_T(\mathbf{t}) = \sum_{i<j} t_i t_j$.

For $k = 1$ we have $\rho(g) = g$, so $\chi_T(\mathbf{t}) = t_1 + t_2 + \cdots + t_n$. For $k = n$, we get the one dimensional representation where $\rho(g)$ is the 1×1-matrix with entry $\det(g)$, so we have $\chi_T(\mathbf{t}) = t_1 t_2 \cdots t_n$. The latter gives the trivial representation when restricted to $\mathrm{SL}(n)$.

Let ρ be any representation of $\mathrm{GL}(n)$. We fix the lexicographic order on the set of weights $\mathbf{b}$ that occur in ρ. Of particular importance is *the highest weight*. The corresponding eigenvectors $w \in W_{\mathbf{b}}$ in (3) are called *highest weight vectors*. They span the *highest weight space*. In Example 13, the highest weight is $(d, 0, \ldots, 0) \in \mathbb{Z}^n$, and a highest weight vector is the monomial x_1^d. In Example , the highest weight is $(1, \ldots, 1, 0, \ldots, 0)$, and a highest weight vector is $e_1 \wedge \cdots \wedge e_k$. In both cases, the highest weight space is 1-dimensional.

Example 15 (Adjoint representation)**.** The space $V = K^{n \times n}$ of $n \times n$ matrices M forms a representation of $GL(n)$ under the action by conjugation, where $\rho(g)(M) := gMg^{-1}$. This is the *adjoint representation*. The weights, known as *roots* in this case, are t_i/t_j with highest weight $(1, 0, \ldots, 0, -1)$. If we restrict it to $\mathrm{SL}(V)$ we have $t_n^{-1} = \prod_{i=1}^{n-1} t_i$ and the highest weight becomes $(2, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$. Again, the highest weight space is 1-dimensional.

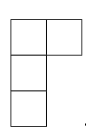
The following proposition provides a characterization of irreducible representations.

Proposition 16. *Every irreducible representation of* $\mathrm{SL}(V)$ *is determined (up to isomorphism) by its highest weight, and the highest weight space is* 1*-dimensional. A weight* $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$ *is the highest weight for some irreducible representation if and only if* $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$.

Proof. For the proof we refer to [1, Chapter 15]. □

Here is a combinatorial tool for building rreducible representations from highest weights:

Definition 17. *A* Young diagram *with k-rows is a nonincreasing sequence of k positive integers. It is usually presented in the following graphical form, e.g. for a sequence* $(2, 1, 1)$*:*



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This particular Young diagram encodes the adjoint representation of $\mathrm{SL}(4)$*.*

Proposition 16 tells us that the irreducible representations of $\mathrm{SL}(n)$ are in bijection with the Young diagrams with at most $n-1$ rows. Representations of $\mathrm{GL}(n)$ are not very different: first, every irreducible representation V of $\mathrm{GL}(n)$ it is also an irreducible representation of $\mathrm{SL}(n)$, so it has a corresponding Young diagram λ. However, different representations of $\mathrm{GL}(n)$ give the same representation of $\mathrm{SL}(n)$ if the differ a power of the determinant. Precisely, consider a representation $\rho : SL(n) \to GL(V)$ with associated Young diagram λ. We have the following representations of $\mathrm{GL}(n)$ for any $a \in \mathbb{Z}$:

$$\rho_a(g) \;:=\; (\det g)^a \cdot \rho\big(\frac{1}{\sqrt[n]{\det g}} \cdot g\,\big).$$

Here, the argument of ρ is in $\mathrm{SL}(n)$. The 1-dimensional representation $g \mapsto \det(g)$ of $\mathrm{GL}(n)$ corresponds to a Young diagram with one column and n rows. Thus for $a \geq 0$ the representation ρ_a corresponds to Young diagram λ extended by a columns of height n. The representation of $\mathrm{GL}(U)$ corresponding to a Young diagram λ is denoted by $S^\lambda(U)$.

Given a Young diagram λ, we write χ_λ for character of the irreducible representation $S^\lambda(U)$. This is a symmetric polynomial in $\mathbf{t} = (t_1, \ldots, t_n)$, known as the *Schur polynomial* of λ. Schur polynomials include the complete symmetric polynomials in Example 13, for $\lambda = (n)$, and the elementary symmetric polynomials in Example 14, for $\lambda = (1, 1, \ldots, 1)$.

Here is an explicit formula for the Schur polynomials, where we set $\lambda_k = 0$ for $k \gg 0$.

Proposition 18. *The Schur polynomial for λ is the following ratio of $n \times n$ determinants:*

$$\chi_\lambda(\mathbf{t}) \quad = \quad \frac{\det\big(\, t_i^{\lambda_j+n-j}\,\big)_{1\leq i,j\leq n}}{\det\big(\, t_i^{n-j}\,\big)_{1\leq i,j\leq n}}.$$

We can find the decomposition (1) of a representation V into irreducibles by writing the character χ_V as linear combination of Schur functions χ_λ with nonnegative integer coefficients a_j. These coefficients are the multiplicities. This expression is unique because the Schur polynomials form a $\mathbb{Z}$-linear basis for the ring of symmetric polynomials in n variables.

Example 19. Let $n = 3$. The Schur polynomial for $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the ternary form

$$\chi_\lambda(\mathbf{t}) \quad = \quad \frac{1}{(t_1-t_2)(t_1-t_3)(t_2-t_3)} \cdot \det \begin{pmatrix} t_1^{\lambda_1+2} & t_1^{\lambda_2+1} & t_1^{\lambda_3} \\ t_2^{\lambda_1+2} & t_2^{\lambda_2+1} & t_2^{\lambda_3} \\ t_3^{\lambda_1+2} & t_3^{\lambda_2+1} & t_3^{\lambda_3} \end{pmatrix}.$$

From this, we compute the three Schur polynomials of degree $|\lambda| = 3$ as follows:

$$\begin{aligned} \chi_{(3,0,0)} &= t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3 + t_1^2 t_3 + t_1 t_2 t_3 + t_2^2 t_3 + t_1 t_3^2 + t_2 t_3^2 + t_3^3 \\ \chi_{(2,1,0)} &= (t_1+t_2)(t_1+t_3)(t_2+t_3) \\ \chi_{(1,1,1)} &= t_1 t_2 t_3 \end{aligned}$$

The action of $\mathrm{GL}(3)$ on $U = K^3$ induces an action on the 27-dimensional space $U^{\otimes 3}$ of $3 \times 3 \times 3$-tensors. As characters are multiplicative under tensor product, its character equals

$$\chi_{U^{\otimes 3}} \;=\; (t_1+t_2+t_3)^3 \;=\; \chi_{(3,0,0)} \,+\, 2 \cdot \chi_{(2,1,0)} \,+\, \chi_{(1,1,1)}.$$

From this decomposition into Schur polynomials, we conclude the irreducible decomposition

$$U^{\otimes 3} \;=\; S^{(3)}(U) \;\oplus\; \big(S^{(21)}(U) \oplus S^{(21)}(U)\big) \;\oplus\; S^{(111)}(U). \tag{4}$$

The first summand is the symmetric tensors, the last summand is the antisymmetric tensors, and the middle summand consists of two copies of the adjoint representation (Example 15).

The irreducible representations $S^\lambda(U)$ of $\mathrm{SL}(U)$ come together with nice algebraic varieties. The group $\mathrm{SL}(U)$ acts also on the projective space $\mathbb{P}(S^\lambda(U))$. The letter action has unique closed orbit, namely the orbit of the highest weight vector. Particular examples are:

1. The orbit of $[e_1 \cdots e_1] \in \mathbb{P}(S^k(U))$. This is the k-th Veronese embedding of $\mathbb{P}(U)$.
2. The orbit of $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\wedge^k(U))$ is the Grassmannian $G(k, U)$ in its Plücker embedding. Here $\lambda = (1, \ldots, 1)$ as in Example 14.

This result provides us with a unified approach to homogeneous varieties. It could be also used to build some of the representations. Fix a Young diagram λ and let $k\lambda$ be a Young diagram where each row is scaled by k. Given the homogeneous variety X in $\mathbb{P}(S^\lambda(U))$ we can take the k-th Veronese map v_k of this projective space and the linear span of $v_k(X)$ is $S^{k\lambda}(V)$. A special case of this construction is point 1 above where $X = \mathbb{P}(U)$.

We shall end this lecture with a beautiful connection between finite groups - S_n and Lie groups - $SL(n)$ or $GL(n)$. This is the *Schur-Weyl* duality. Our reference for this is [1, Chapter 4]. Before stating it let us go back to irreducible representations of S_n. Their characters form a basis of class functions. Hence the number of irreducible representations equals the number of conjugacy classes. Each conjugacy class can be encoded by lengths of cycles in a decomposition of a permutation into cycles. These can be further represented by a Young diagram with n boxes: the first row represents the length of the longest cycle, the last of the shortest. Thus, the number of irreducible representations of S_n equals the number of Young diagrams with n boxes. We shall exhibit a natural bijection between Young diagrams with n boxes and irreducible representations of S_n. Before, we see how to construct it, let us assume that to each such Young diagram λ we can associate a representation S_λ of S_n.

Fix a vector space U and consider the n-fold tensor product $U^{\otimes n}$. There are two groups acting on it: $\mathrm{GL}(U)$ - on each factor - and S_n - by permuting factors. Schur-Weyl duality provides a simultaneous decomposition of the space of tensors with respect to both groups.

Theorem 20 (Schur-Weyl duality)**.** *Let U be a vector space of dimension at least n. Then*

$$U^{\otimes n} \;=\; \sum_{|\lambda|=n} S_\lambda \otimes S^\lambda(U), \tag{5}$$

where the sum is over all Young diagrams with precisely n boxes.

When $n = 2$ we obtain $U^{\otimes 2} = S^2(U) \oplus \bigwedge^2 U$, as there are only two irreducible representations of S_2, both 1-dimensional. This recovers the fact every $n \times n$ matrix is uniquely the sum of a symmetric matrix and a skew-symmetric matrix. The S_2 action on the matrix space $U^{\otimes}$ is transposition, which acts trivially on $S^2(U)$ and changes the sign on $\bigwedge^2 U$.

The case $n = 3$ is the first interesting one. The three irreducible representations S_λ of S_3 in Example 11 correspond to the three outer summands in (4). Note that $\dim(S_\lambda) = 2$ for $\lambda = (2,1)$. The middle summand in (4) is the 16-dimensional space $S_{(21)} \otimes S^{(21)}(U)$.

By Schur-Weyl duality, the multiplicity of $S^\lambda(U)$ in $U^{\otimes n}$ equals the dimension of S_λ. This provides us with a method for defining S_λ. Consider the decomposition of $U^{\otimes n}$ as an $\mathrm{GL}(U)$ representation, into isotypic components. Here the a_λ can be found using Schur functions:

$$U^{\otimes n} \;=\; \oplus_\lambda (S^\lambda(U))^{a_\lambda}.$$

For each isotypic component $(S^\lambda(U))^{a_\lambda}$ consider the highest weight space, i.e. eigenvectors of the torus action with weight λ. The permutation group S_n acts on the highest weight space. This representation of S_n is irreducible, and we find that it is precisely S_λ.

Coming back to the example of matrices ($n = 2$), the highest weight vectors are as follows:

- The highest weight vector $e_1e_1 = e_1 \otimes e_1$ of $S^2(U)$ is invariant with respect to transposition, i.e. it provides the trivial representation of the two-element group S_2.
- The highest weight vector $e_1 \wedge e_2 = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$ of $\wedge^2(U)$ changes sign when transposed, i.e. it provides the sign representation of the two-element group S_2.

Example 21 ($n = 3$)**.** Let $\lambda = (2,1)$. The isotypic component $(S^{(21)}(U))^2$ in the middle of (4) has a 2-dimensional subspace S_λ of highest weight vectors. One possible basis of this space consists of the tensors $e_{112} + e_{211} - e_{121}$ and $e_{121} + e_{211} - e_{112}$, where $e_{ijk} := e_i \otimes e_j \otimes e_k$.

Remark 22. We stress the fact that we worked under the assumption that the field is algebraically closed and of characteristic zero, which makes representation theory much better behaved. Representation theory in finite characteristic is considerably more complicated.

Exercises

1. (a) Prove that, over an algebraically closed field, every irreducible representation of an abelian group is 1-dimensional.

   (b) Explain the correspondence between the characters of a torus $T = (\mathbb{C}^*)^n$, as defined in Lecture 7, and the irreducible representations of T.

2. Derive the character table of the symmetric group S_4. Hint: $1^2+1^2+2^2+3^2+3^2 = 24$. What is the geometric meaning of the 3-dimensional irreducible representations?

3. Let $f : V_1 \to V_2$ be a morphism between two representations of a group G.

   - Prove that the kernel, image and cokernel of f are also representations.
   - Prove that morphisms of two representations are closed under taking scalar multiples and sums, i.e. they form a vector space.

4. Derive the character table of the symmetric group S_5. Hint: $1^2+1^2+4^2+4^2+5^2+5^2+6^2 = 120$. Can you write matrices $\rho(g)$ for the 6-dimensional irreducible representation?

5. Let V_1 and V_2 be two representations of a group G.
   (a) Prove that linear morphisms $\mathrm{Hom}(V_1, V_2)$ have also a structure of a representation. How can you characterize morphisms of representations inside $\mathrm{Hom}(V_1, V_2)$?
   (b) In terms of multiplicities of isotypic components of V_1 and V_2, what is the dimension of the space of morphisms among these two representations?
   (c) Conclude that the multiplicity of an irreducible representation W in V_1 equals the dimension of morphisms of representations $W \to V_1$ (or equivalently of $V_1 \to W$).
6. Let V be a representation of $\mathrm{GL}(n)$. Its character χ_V is a Laurent polynomial in $t_1, \ldots, t_n$. Argue that the vector spaces $S^2(V)$ and $\bigwedge^2 V$ are also representations of $\mathrm{GL}(V)$, and compute the characters $\chi_{S^2(V)}$ and $\chi_{\bigwedge^2 V}$ in terms of χ_V.
7. Describe the 2-dimensional irreducible representation from Example 11 explicitly, by assigning a 2×2 matrix to each of the six permutations of the set $\{1, 2, 3\}$.
8. Consider the representation ρ of $\mathrm{GL}(3)$ action on $\bigwedge^3 K^6$? What is the highest weight? What is the associated Young diagram? Find the entries of the 20×20-matrix $\rho(g)$.
9. Is every $2 \times 2 \times 2$ tensor the sum of a symmetric and a skew-symmetric tensor?
10. If $U = K^n$, what is the dimension of $S^{(2,1)}(U)$? Give a formula in terms of n.
11. What is the dimensions of the vector space $S^3(S^3(K^3))$? Find a weight basis. Write the character of this $\mathrm{GL}(3)$ representation. Can you decompose it into Schur polynomials?
12. What are the orbits of the adjoint representation? Are they closed? What is the dimension of a general orbit? What is the vanishing ideal such an orbit, e.g. for $n = 3$?

References

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