

Polytopes and Matroids

Notes by Mateusz Michałek
for the lecture on July 10, 2018, in the
IMPRS Ringvorlesung *Introduction to Nonlinear Algebra*

In the last lecture we discuss relations among (lattice) polytopes and matroids. As we have seen in Lecture 7 lattice polytopes are strongly related to toric varieties. Indeed, toric geometry has many connections to the theory of matroids and in this lecture we will be able to present just a few of them. For many more interesting results we refer to [2, 5, 3, 6]. Our main aim will be to show connections among: the Grassmannians (Lecture 4), toric varieties (Lecture 7) and matroids.

We start by recalling the definition of a lattice polytope.

Definition 1 (Lattice polytope). *Let \mathbb{R}^n be a real vector space. A polytope P is the convex hull of a finite set of points $p_1, \dots, p_k \in \mathbb{R}^n$:*

$$P := \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^k \lambda_i p_i \text{ for some real } \lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}.$$

We say that P is a lattice polytope if we may find $p_1, \dots, p_k \in \mathbb{Z}^n \subset \mathbb{R}^n$.

For each polytope P there is an inclusion minimal set of p_i 's of which it is a convex hull. We call these p_i 's the vertices of P .

The second basic class of objects we discuss are matroids. Their name suggests that they could be regarded as generalizations of matrices. Indeed, as we will soon see every matrix defines a matroid. Just as the groups abstract the notion of symmetry, matroids abstract the notion of independence. We fix a finite set E , which we will refer to as the ground set of a matroid. We would like to distinguish a family of subsets of E that we could call independent. Thus a matroid M will be a family $\mathfrak{I} \subset 2^E$ of subsets E that we refer to as independent sets. These are of course assumed to satisfy certain axioms.

A first observation is that whenever we have an independent set $I \subset E$, it is reasonable to assume that every subset of I is also independent. We obtain the first axiom of a matroid for the family \mathfrak{I} :

1. If $I \in \mathfrak{I}$ and $J \subset I$, then $J \in \mathfrak{I}$.

What we defined so far is a very important object in mathematics: *simplicial complex*. Another observation is that we would like \mathfrak{I} to be nonempty, or equivalently we want the empty set to be independent:

2. We have $\emptyset \in \mathfrak{I}$.

It turns out that to obtain a matroid we need just one more axiom. To motivate it we make a following observation. Whenever we have finite linearly independent subsets $I, J \subset V$ of a vector space V , if $|I| < |J|$, then we may extend I by an element of J , in such a way that $I \cup \{j\}$ is still linearly independent. This simple observation is precisely what we need to get the last axiom for the family \mathfrak{I} :

3. If $I, J \in \mathfrak{I}$ and $|I| < |J|$ then there exists $j \in J$ such that $I \cup \{j\} \in \mathfrak{I}$.

Definition 2. A matroid is a family of subsets \mathfrak{I} satisfying Axioms 1,2 and 3 above.

In Exercise 1 the reader is asked to prove that the following structures are matroids.

Example 3. • (Representable/Realizable matroid) Let V be a vector space over an arbitrary field F . Let $E \subset V$ be a nonempty, finite subset. We define \mathfrak{I} to be the family of subsets of E that are linearly independent. We say that the matroid is representable over F .

- (Graphic matroid) Let G be a graph with edge set E . Let \mathfrak{I} be the family of those subsets of E that do not contain a cycle. Equivalently \mathfrak{I} is the family of forests in G .
- (Algebraic matroid) Let $F \subset K$ be an arbitrary field extension. Let E be a finite subset of K . Let \mathfrak{I} be the family of subsets of E that are algebraically independent over F .
- (Uniform matroid) Let E be a finite set and $k \leq |E|$. Let \mathfrak{I} be the family of subsets of cardinality at most k . This matroid is denoted by $U_{k,E}$ or $U_{k,|E|}$.

Matroids are known for having many equivalent definitions, depending on the point of view on the matroid. For example, due to the first axiom to determine a matroid we do not have to know all independent sets, just those that are inclusion maximal. By analogy to linear algebra, the inclusion maximal independent sets are called *basis*. It turns out - as the reader is asked to prove in Exercise 2 - that a nonempty family $\mathfrak{B} \subset 2^E$ of subsets of E is a family of basis of some matroid if and only if the following axiom is satisfied:

- For all $B_1, B_2 \in \mathfrak{B}$, $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\} \in \mathfrak{B}$.

The seemingly weak axiom on \mathfrak{B} in fact implies the following two statements:

- For all $B_1, B_2 \in \mathfrak{B}$, $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that both $(B_1 \setminus \{b_1\}) \cup \{b_2\}, (B_2 \setminus \{b_2\}) \cup \{b_1\} \in \mathfrak{B}$.
- For all $B_1, B_2 \in \mathfrak{B}$ and any subset $A_2 \subset B_2 \setminus B_1$ there exists a subset $A_1 \subset B_1 \setminus B_2$ such that both $(B_1 \setminus A_1) \cup A_2, (B_2 \setminus A_2) \cup A_1 \in \mathfrak{B}$.

The first point is known as the *symmetric exchange property* and the second one as *multiple symmetric exchange property*. The facts that both exchange properties hold is nontrivial - we refer the reader to the proofs in [1, 8]. We will soon see the algebraic meaning of the exchange properties.

Exercise 3 states that all basis of a matroid have the same cardinality. The cardinality of a basis is known as the *rank* of a matroid. More generally for a matroid on a ground set E we may define the rank of any subset of $A \subset E$.

Definition 4. For a matroid on a ground set E and independent sets $\mathfrak{I} \subset 2^E$ we define the rank function:

$$r : 2^E \ni A \rightarrow \max_{I \in \mathfrak{I}} \{|I \cap A|\} \in \mathbb{Z}.$$

Equivalently, the rank of a set is the cardinality of a largest independent set contained in it.

We note that for a representable matroid the rank is simply the dimension of the vector subspace spanned by the given vectors. Clearly, for any matroid the rank function r satisfies the following:

- $0 \leq r(A)$ for all $A \subset E$ and $r(\emptyset) = 0$.
- $r(A) \leq r(A \cup \{b\}) \leq r(A) + 1$ for all $A \subset E, x \in E$.

Further, the rank function has one more property known as *submodularity*:

- for all $A, B \subset E$ we have $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

In Exercise 7 the reader is asked to prove that any function $r : 2^E \rightarrow \mathbb{Z}$ satisfying the three axioms above is a rank function of a matroid. The independent sets can be reconstructed as those $I \subset E$ for which $r(I) = |I|$. This gives us another possible definition of a matroid.

To pass from a combinatorial object, like a matroid, to a polytope, we apply the following 'standard' construction. Consider a vector space $\mathbb{R}^{|E|}$ with basis elements b_e corresponding to the elements $e \in E$. Any subset $A \subset E$ can be identified with a point $p_A := \sum_{e \in A} b_e \in \mathbb{R}^{|E|}$. In this way a family of subsets may be identified with a set of points.

Definition 5 (Matroid basis polytope). Let M be a matroid on the ground set E and basis set \mathfrak{B} . We use the notation introduced above. We define the matroid basis polytope $P_M \subset \mathbb{R}^{|E|}$ as the convex hull of the points $p_B := \sum_{e \in B} b_e \in \mathbb{R}^{|E|}$, where we take all $B \in \mathfrak{B}$.

Clearly P_M is a lattice polytope, hence we may consider the toric variety associated to it. Precisely it is the image of the map given by monomials, in variables corresponding to elements of E , that are products of elements in a basis.

Example 6. Consider a rank two uniform matroid on the set $E = \{1, 2, 3\}$. Precisely:

$$\mathfrak{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

We consider \mathbb{R}^3 . The three basis above correspond, in the given order to the three points:

$$(1, 1, 0), (1, 0, 1), (0, 1, 1) \in \mathbb{R}^3.$$

Hence, the matroid basis polytope is a two dimensional triangle. The polynomial map is:

$$(\mathbb{C}^*)^3 \ni (x_1, x_2, x_3) \rightarrow (x_1x_2, x_1x_3, x_2x_3) \in \mathbb{P}^2.$$

The closure of the image is the whole \mathbb{P}^2 , which is the associated toric variety.

The combinatorial statement equivalent to the proposition presented below was proved by White [7].

Proposition 7. *A matroid basis polytope is normal in the lattice that it spans.*

In order to present the proof we state one of the most useful theorems about matroids.

Theorem 8 (The matroid union theorem). *Let M_1, \dots, M_k be matroids on the same ground set E with respective families of independent sets $\mathfrak{I}_1, \dots, \mathfrak{I}_k$ and rank functions r_1, \dots, r_k . Let*

$$\mathfrak{I} := \{I \subset E : I = \bigcup_{i=1}^k I_i \text{ for } I_i \in \mathfrak{I}_i\}.$$

Then \mathfrak{I} is also a family of independent sets for a matroid, known as the union of M_1, \dots, M_k . Further, the rank of any set $A \subset E$ for the union matroid is given by:

$$r(A) = \min_{B \subset A} \{|A \setminus B| + \sum_{i=1}^k r_i(B)\}.$$

For the proof we refer to [6, 12.3.1]. As a corollary of the matroid union theorem we obtain the following theorem due to Edmonds.

Theorem 9. *Let M be a matroid on a ground set E with rank function r . E can be partitioned into k independent sets if and only if $|A| \leq k \cdot r(A)$ for all subsets $A \subset E$.*

Proof. The implication \Rightarrow is straightforward.

For the other implication consider the union U of M with itself k times. We apply the matroid union theorem to compute the rank of E :

$$r_U(E) = \min\{|E| - |B| + k \cdot r_M(B)\}.$$

Clearly by assumption $|E| - |B| + k \cdot r_M(B) \geq |E|$ and equality holds for $B = \emptyset$. Hence, $r_U(E) = |E|$. This means that E is an independent set in U , and hence by definition it is a union of k independent sets of M . \square

Definition 10. *Let M be a matroid on a ground set E with the family of independent sets \mathfrak{I} . Let $E' \subset E$. The restriction of M to E' is a matroid where $A \subset E'$ is independent if and only if $A \in \mathfrak{I}$.*

Proof of Proposition 7. Let M be a matroid on the ground set $\{1, \dots, n\}$. Let $p \in kP_M$. We know that $p = \sum_{b \in \mathfrak{B}} \lambda_B p_B$ with $\sum \lambda_B = k$ and $0 \leq \lambda_B \in \mathbb{Q}$. After clearing the denominators we have:

$$dp = \sum \lambda'_B p_b,$$

where $\sum \lambda_B = dk$ and $0 \leq \lambda_B \in \mathbb{Z}$.

By restricting the matroid M we may assume that all coordinates of $p = (p_1, \dots, p_n)$ are nonzero.

We define two matroids. The first matroid N is on the ground set $E_N := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq p_i\}$. In other words, we replace a point i in the original matroid by p_i equivalent points. A subset $\{(i_1, j_1), \dots, (i_s, j_s)\} \subset E_N$ is independent if only if:

- all i_q 's are distinct,
- $\{i_1, \dots, i_s\}$ is an independent set in M .

We note that a basis of N maps naturally to a basis of M . Also the rank function for N is the same as the one for M if we forget the second coordinates. Further, the point p has a decomposition as a sum of k points corresponding to basis of M if and only if the matroid N is covered by k basis (i.e. the ground set is a union of k basis). Hence, by Theorem 9 our aim is to prove the following statement:

For any $A \subset E_N$ we have $|A| \leq kr_N(A)$.

The second matroid N' is on the ground set $E_{N'} := \{(i, j, l) : 1 \leq i \leq n, 1 \leq j \leq p_i, 1 \leq l \leq d\}$. In other words we replace any point of N by d equivalent points. A subset $\{(i_1, j_1, l_1), \dots, (i_s, j_s, l_s)\} \subset E_{N'}$ is independent if only if:

- all i_q 's are distinct,
- $\{i_1, \dots, i_s\}$ is an independent set in M .

We have a natural projection $\pi : E_{N'} \rightarrow E_N$ given by forgetting the last coordinate. We note that $r_{N'}(\pi^{-1}(A)) = r_N(A)$. As the point dp is decomposable we know that the matroid N' can be covered by kd basis. Hence, for any $B \subset E_{N'}$ we have: $|B| \leq dk \cdot r_{N'}(B)$. Applying this to $\pi^{-1}(A)$ we obtain:

$$k|A| = |\pi^{-1}(A)| \leq dk \cdot r_{N'}(\pi^{-1}(A)) = dk \cdot r_N(A).$$

This is equivalent to the statement we wanted to prove! □

Our next aim is to relate matroids with the geometry of special subvarieties of Grassmannians. We recall that one of the possible definitions of a Grassmannian $G(k, n)$ is an orbit of $[e_1 \wedge \dots \wedge e_k] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ under the action of the group of $n \times n$ invertible matrices $GL(n)$. While the Grassmannian is an orbit of the big group $GL(n)$, we may ask how smaller groups act on $G(k, n)$. In particular, consider the torus $T := (\mathbb{C}^*)^n$ of diagonal matrices. This torus acts on $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ and on $G(k, n)$. However, in general $G(k, n)$ is not an orbit of T or even a closure of an orbit of T . Indeed, we already know that $G(k, n)$ has dimension $k(n - k)$ which may be much larger than n . Let us fix a point $p \in G(k, n)$. The questions that motivate us are as follows:

- What is the T -orbit of p ?
- What is the closure of this orbit?
- How can we describe this variety?

A beautiful answer was provided by Gelfand, Goresky, MacPherson and Serganova [4]. The point $p = [v_1 \wedge \cdots \wedge v_k] \in G(k, n)$ represents a k -dimensional subspace $V = \langle v_1, \dots, v_k \rangle$ in \mathbb{C}^n . We may present the vectors v_1, \dots, v_k as a $k \times n$ matrix N_p . From Lecture 4 we know that the coordinates of $p \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$, are given by maximal minors of N_p . How does a point $t = (t_1, \dots, t_n) \in T$ act on p ? In general, t acts on the coordinate indexed by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ rescaling it by $t_{i_1} \cdots t_{i_k}$. Hence, the orbit of p is the image of the map:

$$T \ni (t_1, \dots, t_n) \rightarrow (t_{i_1} \cdots t_{i_k} \det((N_p)_{i_1, \dots, i_k}))_{1 \leq i_1 < \dots < i_k \leq n} \in \mathbb{P}(\bigwedge^k \mathbb{C}^n),$$

where $(N_p)_{i_1, \dots, i_k}$ denotes the $k \times k$ submatrix of N_p with the chosen columns indexed by i_1, \dots, i_k .

Example 11. Consider the two dimensional subspace of the four dimensional space spanned by the rows of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

In the coordinates of the Grassmannian we have the associated point:

$$(e_1 + e_2 + e_3 + e_4) \wedge (e_1 + 2e_2 + 3e_3 + 4e_4) = e_1 \wedge e_2 + 2e_1 \wedge e_3 + 3e_1 \wedge e_4 + e_2 \wedge e_3 + 2e_2 \wedge e_4 + e_3 \wedge e_4.$$

The orbit in the coordinates above is parameterized as follows:

$$(t_1, t_2, t_3, t_4) \rightarrow (t_1 t_2, 2t_1 t_3, 3t_1 t_4, t_2 t_3, 2t_2 t_4, t_3 t_4).$$

We see that the orbit is almost an image of a monomial map! Indeed, the only thing that changes are the constants given by minors of the matrix N_p . However, these constants do not depend on $t \in T$ and hence we may define an automorphism of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ that turns the orbit to an image of a monomial map, by simply rescaling the coordinates.

At this point one could have a false impression that the orbit is isomorphic to the image of a monomial map defined by all squarefree monomials of degree k . This is not the case, as some of the monomials may not appear at all! This happens if the corresponding minor was equal to zero - then we cannot rescale it.

Example 12. 1. First we continue Example 11. The polytope associated to the toric variety has the following vertices:

$$(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1).$$

This is the hypersimplex $\Delta_{2,4}$. The associated projective toric variety is three dimensional. It represents the closure of the T -orbit of a general point in the Grassmannian $G(2,4)$. The associated matroid is the uniform rank two matroid on four elements.

2. Let us now consider a different point of $G(2,4)$ given by the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

The orbit is parameterized as follows:

$$(t_1, t_2, t_3, t_4) \rightarrow (2t_1t_2, 3t_1t_3, 4t_1t_4, 0, 0, 0).$$

The polytope representing the toric variety has the following vertices:

$$(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1).$$

It is isomorphic to a two dimensional simplex, hence the closure of the orbit is a \mathbb{P}^2 , as can be seen directly from the parameterization.

Which monomials are thus left? Exactly those for which the corresponding minor of N_p was not zero.

Let us consider the representable matroid M_p of n points in \mathbb{C}^k , defined by the columns of the matrix N_p . Clearly a set of points is a basis of M_p if and only if the corresponding minor of N_p is nonzero. We have proved the following proposition.

Proposition 13. *The closure of the T -orbit of any point $p = [v_1 \wedge \dots \wedge v_k]$ in a Grassmannian $G(k, n)$ is the toric variety represented by the matroid base polytope, for the representable matroid defined by columns of the $k \times n$ matrix N_p with i -th row equal to v_i .*

The results of Lecture 7 combined with Proposition 7 show the following.

Proposition 14. *Any torus orbit closure in any Grassmannian is projectively normal.*

We now turn to the interpretation of basis exchange properties in terms of algebraic geometry. Consider a matroid with basis polytope P . We recall that:

- the ideal of the associated toric variety is generated by binomials,
- every binomial in the ideal corresponds to an integral relation among lattice points of P .

How do these statements specialize in the case of matroids? A lattice point of P is the characteristic function of a basis. A sum of lattice points is the sum of these characteristic functions. This corresponds to taking a sum of basis as *multisets*.

Example 15. *Consider the rank two uniform matroid on four elements $\{p_1, p_2, p_3, p_4\}$. An integral relation among the lattice points of the basis polytope is:*

$$(1, 1, 0, 0) + (0, 0, 1, 1) = (1, 0, 1, 0) + (0, 1, 0, 1).$$

As a sum of basis elements this corresponds to:

$$\{p_1, p_2\} \cup \{p_3, p_4\} = \{p_1, p_3\} \cup \{p_2, p_4\}.$$

It is a degree two binomial in the ideal of the associated toric variety.

Hence, we say that two multisets of basis are *compatible* if their union (as multisets) is the same. Equivalently, every element of the base set belongs to the same number of basis in the first and second multiset of basis. Thus, the binomials in the ideal of the toric variety represented by matroid base polytope are in bijection with pairs of compatible multisets of basis.

What are the quadrics in such an ideal? Equivalently, when $\{B_1, B_2\}$ is equivalent to $\{B_3, B_4\}$? This is if and only if $B_1 \cup B_2 = B_3 \cup B_4$. In other words, this is if and only if we change:

- B_1 by subtracting from it a set $A_1 \subset B_1 \setminus B_2$ and adding to it $A_2 \subset B_2 \setminus B_1$ and
- B_2 by adding to it A_1 and subtracting A_2 .

We see that quadrics in the ideal correspond to multiple symmetric exchanges. It follows that symmetric basis exchanges form a distinguished set of quadrics in the ideal. The following four conjectures are due to White.

Conjecture 16. • *Representable case: The ideal of any torus orbit closure in any Grassmannian is:*

1. *generated by quadrics,*
 2. *generated by quadrics corresponding to symmetric basis exchanges.*
- *General case: For any matroid M any two finite multisets of basis $(B_i), (B_j)$ such that $\bigcup B_i = \bigcup B_j$ can be transformed to one another in a finite number of such steps that:*
1. *we replace two basis B, B' in one multiset, by two basis \tilde{B}, \tilde{B}' obtained by multiple symmetric exchange (i.e. $B \cup B' = \tilde{B} \cup \tilde{B}'$),*
 2. *we replace two basis B, B' in one multiset, by two basis \tilde{B}, \tilde{B}' obtained by a symmetric exchange (i.e. $B = \tilde{B} \cup \{b_1\} \setminus \{b_2\}$ and $B' = \tilde{B}' \cup \{b_2\} \setminus \{b_1\}$).*

It is an easy exercise to show that the general case implies the representable case.

Exercises

1. Show that Example 3 presents matroids.
2. a) Fix a family of independent sets \mathfrak{I} for a matroid M . Prove that the inclusion maximal elements in \mathfrak{I} satisfy the axiom for the basis of a matroid.
b) Fix a nonempty set $\mathfrak{B} \subset 2^E$ satisfying the axiom for basis of a matroid. Prove that $\mathfrak{I} := \{I \subset E : \exists_{B \in \mathfrak{B}} : I \subset B\}$ satisfies the axioms for the independent sets.
3. Prove that all basis in a matroid have the same cardinality.
4. Prove that the points p_B in Definition 5 are vertices of the polytope P_M . Prove that these are the only lattice points of P_M .

5.
 - Let $\mathfrak{B} \subset 2^E$ be a set of basis of a matroid M . Let $\mathfrak{B}^* := \{B \subset E : E \setminus B \in \mathfrak{B}\}$. Prove that \mathfrak{B}^* is a set of basis of a matroid M^* . The matroid M^* is known as the dual matroid (of M).
 - Prove that a dual of a representable matroid is representable.
6. Prove that for any matroid the rank function is submodular.
7. Prove that any function $2^E \rightarrow \mathbb{Z}$ satisfying the three axioms of the rank function is indeed a rank function of some matroid.
8. How many distinct torus orbit closures are there in $G(2, 4)$? How many up to isomorphism (of algebraic varieties)?
9. Prove White's conjectures for uniform matroids.

References

- [1] R.A. Brualdi: *Comments on bases in dependence structures*, Bull. Austral. Math. Soc. 1 (1969), 161-167.
- [2] E.M Feichtner, B. Sturmfels: *Matroid polytopes, nested sets and Bergman fans*. Portugaliae Mathematica 62.4 (2005): 437-468.
- [3] A.Fink, D.E. Speyer: *K-classes for matroids and equivariant localization*. Duke Mathematical Journal 161.14 (2012): 2699-2723.
- [4] I.M. Gelfand, R. Goresky, R.D. MacPherson, V.V. Serganova: *Combinatorial geometries, convex polyhedra, and Schubert cells*. Advances in Mathematics, 63(3) (1987): 301-316.
- [5] J. Huh: *h-Vectors of matroids and logarithmic concavity*. Advances in mathematics 270 (2015): 49-59.
- [6] J.G. Oxley: *Matroid Theory*, Oxford Science Publications, Oxford University Press, Oxford, 1992.
- [7] N. White: *The basis monomial ring of a matroid*, Adv. Math. 24 (1977), 292-297.
- [8] D.R. Woodall: *An exchange theorem for bases of matroids*, J. Combin. Theory Ser. B 16 (1974), 227-228.