Elimination and Implicitization

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We fix an algebraically closed field K and the polynomial ring $K[\mathbf{x}] = K[x_1, \ldots, x_n]$. Every ideal $I \subset K[\mathbf{x}]$ has an associated variety $\mathcal{V}(I) = \{ \mathbf{p} \in K^n : f(\mathbf{p}) = 0 \text{ for all } f \in I \}$. Consider the projection from K^n onto the subspace given by the first m coordinates:

$$\pi : K^n \to K^m,\ (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) \mapsto (p_1, \ldots, p_m).$$

If V is a variety in K^m then its image $\pi(V)$ need not be a variety.

Example 1 ($n = 2, m = 1$)**.** The image of the hyperbola $V = \mathcal{V}(xy-1)$ under the projection $K^2 \to K^1$ from the plane to the x-axis equals $\pi(V) = K^1 \backslash \{0\}$. This is not a variety in K^1.

By definition, the Zariski closure $\overline{\pi(V)}$ of the image $\pi(V)$ is a variety in K^m. We call $\overline{\pi(V)}$ the *closed image* of V under the map π. The following theorem characterizes its ideal.

Theorem 2. *Let $V = \mathcal{V}(I)$ be the variety given by the ideal $I \subset K[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$. Then its closed image in K^m is the variety $\overline{\pi(V)} = \mathcal{V}(J)$ defined by the* elimination ideal

$$J \;=\; I \cap K[x_1, \ldots, x_m]. \tag{1}$$

Proof. See [1, §2.2, Theorem 3]. □

Theorem 2 says that the algebraic operation of elimination corresponds to the geometric operation of projection. This holds in many settings, not just in algebraic geometry. For instance, Gaussian elimination in linear algebra corresponds to projection of linear subspaces, and Fourier-Motzkin elimination in convex geometry corresponds to projection of polyhedra.

Example 3 (Matrix Completion)**.** Fix $n = 15$ and let V be the variety of symmetric 5×5-matrices $X = (x_{ij})$ of rank ≤ 2. Its ideal $I = \mathcal{I}(V)$ is minimally generated by 50 homogeneous cubic polynomials, namely the 3×3-minors of X. These cubics from a Gröbner basis for the degree reverse lexicographic order. Now let $m = 10$ and order the variables so that the five diagonal entries $x_{11}, x_{22}, x_{33}, x_{44}, x_{55}$ come last. Then the elimination ideal is principal:

$$\begin{aligned} J \quad = \quad & \langle\, x_{14}x_{15}x_{23}x_{25}x_{34} - x_{13}x_{15}x_{24}x_{25}x_{34} - x_{14}x_{15}x_{23}x_{24}x_{35} + x_{13}x_{14}x_{24}x_{25}x_{35} \\ & + x_{12}x_{15}x_{24}x_{34}x_{35} - x_{12}x_{14}x_{25}x_{34}x_{35} + x_{13}x_{15}x_{23}x_{24}x_{45} - x_{13}x_{14}x_{23}x_{25}x_{45} \\ & - x_{12}x_{15}x_{23}x_{34}x_{45} + x_{12}x_{13}x_{25}x_{34}x_{45} + x_{12}x_{14}x_{23}x_{35}x_{45} - x_{12}x_{13}x_{24}x_{35}x_{45} \,\rangle. \end{aligned}$$

The ideal generator is known as the *pentad* in algebraic statistics [3, Example 4.2.8]. The 15 terms correspond to the 15 maximal matchings in the complete graph K_5. The hypersurface $\mathcal{V}(J)$ equals the image $\pi(V)$ of the determinantal variety V under the projection onto the K^{10} given by the off-diagonal entries. If the 10 off-diagonal entries of a symmetric 5×5-matrix are given then that matrix can be completed to a matrix of rank ≤ 3 if and only if the pentad vanishes. This constraint appears in the statistical theory of *factor analysis* [3]. It represents a widely studied class of problems known as *(low rank) matrix completion*.

Example 4. The first four power sums in three variables are $x^i + y^i + z^i$ for $i = 1, 2, 3, 4$. These must be algebraically dependent. But, what is the algebraic relation satisfied by these power sums? We approach this question by setting $n = 7, m = 4$ and introducing the ideal

$$I \;=\; \langle\, x+y+z-p_1,\; x^2+y^2+z^2-p_2,\; x^3+y^3+z^3-p_3,\, x^4+y^4+z^4-p_4 \,\rangle.$$

This ideal lives in a polynomial ring in 7 variables. We are interested in its elimination ideal

$$J \;=\; I \;\cap\; K[p_1, p_2, p_3, p_4].$$

This is a principal prime ideal. Its generator has degree 4. This gives the desired relation:

$$J \;=\; \langle\, p_1^4 - 6p_1^2 p_2 + 3p_2^2 + 8p_1 p_3 - 6p_4 \,\rangle.$$

The computations in our two examples were carried out using Gröbner bases. Here is how this works. We first fix the lexicographic monomial order $\prec$ on $K[\mathbf{x}]$ with $x_1 \prec x_2 \prec \cdots \prec x_n$. We then compute the reduced Gröbner basis for the ideal generated by the given polynomials. And, finally, we select those polynomials from the output that use only the first m variables.

Theorem 5. *If $\mathcal{G}$ is a lexicographic Gröbner basis for an ideal I in $K[\mathbf{x}]$ then its elimination ideal (1) has the Gröbner basis $\mathcal{G}' = \mathcal{G} \cap K[x_1, \ldots, x_m]$. If $\mathcal{G}$ is reduced then so is $\mathcal{G}'$.*

Proof. Clearly, $\mathcal{G}'$ is contained in $J = I \cap K[x_1, \ldots, x_m]$. Consider any nonzero polynomial $f \in J$. The initial monomial $\mathrm{in}_\prec(f)$ is divisible by $\mathrm{in}_\prec(g)$ for some $g \in \mathcal{G}$. None of the variables $x_{m+1}, \ldots, x_n$ appears in the monomial $\mathrm{in}_\prec(g)$. Every trailing term of g is lexicographically smaller, so it cannot use any of the last $n - m$ variables. Hence g lies in $\mathcal{G}'$. We have shown that some initial monomial from $\mathcal{G}'$ divides $\mathrm{in}_\prec(f)$. Since f was chosen arbitrarily from $J \backslash \{0\}$, this means that $\mathcal{G}'$ is a Gröbner basis for J. If the given Gröbner basis $\mathcal{G}$ is reduced then $\mathcal{G}'$ also satisfies the two requirements for being a reduced Gröbner basis. □

This result shows that the lexicographic Gröbner basis $\mathcal{G}$ solves the elimination problem simultaneously for all m. Thus computing $\mathcal{G}$ means triangularizing a given system of polynomial equations. We saw in Lecture 1 that it can be quite costly to compute a lexicographic Gröbner basis. One therefore often uses different strategies to carry out the elimination process. But Theorem 5 represents the main idea that underlies these strategies.

Example 6. Are there real numbers x, y, z whose i-th power sum equals i for $i = 1, 2, 3$? To answer this question, we compute the lexicographic Gröbner basis of the ideal

$$I \;=\; \langle x+y+z-1,\, x^2+y^2+z^2-2,\, x^3+y^3+z^3-3 \,\rangle.$$

This Gröbner basis equal

$$\mathcal{G} \;=\; \{\, 6\underline{z^3} - 6z^2 - 3z - 1,\ 2\underline{y^2} + 2yz - 2y + 2z^2 - 2z - 1,\ \underline{x} + y + z - 1 \,\}.$$

Theorem 2 says that we can solve our equations by back-substitution. We first compute the three roots of the cubic in z, we substitute them into the second equation and solve for y, and then we set $x = 1 - y - z$. The cubic has one real root and two complex conjugate roots:

$$z \;\in\; \{1.4308, -0.21542 - 0.26471i, -0.21542 + 0.26471i\}.$$

Hence the answer to our question is "no". The variety $\mathcal{V}(I)$ has no real points.

Implicitization is a special instance of elimination. Here, the problem is to compute the image of a polynomial map. This can be done by forming the graph of the map and then projecting onto the image coordinates. To be precise, we consider a map of the form

$$f \,:\, K^m \to K^n,\ \mathbf{p} = (p_1, \ldots, p_m) \mapsto \big(f_1(\mathbf{p}), \ldots, f_n(\mathbf{p})\big),$$

where $f_1, \ldots, f_n$ are polynomials in $K[z_1, \ldots, z_m]$. We write image(f) for the image of K^m under this map. This need not be a variety, as the following example shows:

Example 7. Let $m = 2, n = 3$ and consider the map given by $(z_1, z_1z_2, z_1z_2^2)$. The Zariski closure of the image is the surface $V = \mathcal{V}(x_1x_3 - x_2^2)$ in K^3. The point $(0, 0, 1)$ lies on this surface but it is not in image(f). For $K = \mathbb{C}$ we can approximate $(0, 0, 1)$ by a sequence of points in the image, namely by taking $z_1 = \epsilon^2$ and $z_2 = \epsilon^{-1}$ for $\epsilon \to 0$.

Recall that the closed image of the map $f : K^m \to K^n$ is the Zariski closure of the set-theoretic image image(f). The closed image is denoted $\overline{\text{image}}(f)$.

Corollary 8. *Let I be the ideal in the polynomial ring $K[\mathbf{x}, \mathbf{z}]$ in $n + m$ variables which is generated by $f_i(z_1, \ldots, z_m) - x_i$ for $i = 1, 2, \ldots, n$. The closed image of $f : K^m \to K^n$ is the variety defined the elimination ideal $J = I \cap K[\mathbf{x}]$. In symbols, $\overline{\text{image}}(f) = \mathcal{V}(J)$.*

Proof. The graph of f is Zariski closed in K^{n+m}, and I is the ideal that defines it. The image of f is the projection of the graph onto K^n. With this, the claim follows from Theorem 2. □

Example 9 (Plücker relations)**.** What are the algebraic relations among the 2×2-minors of a 2×5-matrix? We answer this question by setting $m = n = 10$ and considering the map $f : K^{10} \to K^{10}$ that takes a matrix $\begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \end{pmatrix}$ to the vector $(x_{12}, x_{13}, \ldots, x_{45})$ where $x_{ij} = z_{1i}z_{2j} - z_{1j}z_{2i}$ for $1 \le i < j \le 5$. The graph of f is described by an ideal I in the polynomial ring $K[\mathbf{x}, \mathbf{z}]$ in 20 variables. The desired elimination ideal equals

$$\begin{aligned} I \cap K[\mathbf{x}] \;=\; & \langle\, x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}\,,\ x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}\,, \\ & x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24},\ x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34},\ x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} \,\rangle. \end{aligned}$$

These quadrics are the Plücker relations among the maximal minors. They will play a key role in our study of Grassmannians in the next lecture. Consider the skew-symmetric matrix

$$X \;=\; \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}.$$

The Plücker relations are the *pfaffians* of size 4×4, that is, the square roots of the principal 4×4 minors of X. Thus $\mathcal{V}(I \cap K[\mathbf{x}])$ is the variety of skew-symmetric 5×5 matrices of rank ≤ 2. We shall see that, as a projective variety in $\mathbb{P}^9$, this is the Grassmannian of lines in $\mathbb{P}^4$.

Example 10 (Hyperdeterminant)**.** Let $X = (x_{ijk})$ be a tensor of format $2 \times 2 \times 2$. Its entries are $n = 8$ variables. The tensor represents a trilinear polynomial in $m = 3$ variables:

$$f \;=\; x_{000} + x_{100}z_1 + x_{010}z_2 + x_{001}z_3 + x_{110}z_1z_2 + x_{101}z_1z_3 + x_{011}z_2z_3 + x_{111}z_1z_2z_3.$$

The surface $\mathcal{V}(f)$ is singular at the point $\mathbf{z}$ if and only if the pair $(X, \mathbf{z})$ lies in the variety of

$$I \;=\; \langle\, f, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \,\rangle.$$

The elimination ideal $I \cap K[\mathbf{x}]$ is principal. We find that its generator is the quartic

$$x_{110}^2x_{001}^2 + x_{100}^2x_{011}^2 + x_{010}^2x_{101}^2 + x_{000}^2x_{111}^2 + 4x_{000}x_{110}x_{011}x_{101} + 4x_{010}x_{100}x_{001}x_{111} - 2x_{100}x_{110}x_{001}x_{011}$$
$$-2x_{010}x_{110}x_{001}x_{101} - 2x_{010}x_{100}x_{011}x_{101} - 2x_{000}x_{110}x_{001}x_{111} - 2x_{000}x_{100}x_{011}x_{111} - 2x_{000}x_{010}x_{101}x_{111}.$$

This is the $2 \times 2 \times 2$ *hyperdeterminant*. It vanishes whenever the surface $V(f)$ fails to be smooth in K^3. The study of hyperdeterminants is a fascinating topic in nonlinear algebra.

The most basic scenario in elimination arises when m variables are eliminated from a system of $m+1$ equations to yield a single polynomial in the coefficients of that system. We saw this for $m = 3$ in Examples 4 and 10. The theory of *resultants* is custom-taylored to predict the eliminant in such scenarios. We set this up as follows.

Let $i \in \{1, 2, \ldots, m+1\}$ and fix a general polynomial f_i of degree d_i in $z_1, \ldots, z_m$. This polynomial has $\binom{d_i+m}{m}$ unknown coefficients $x_{i,\mathbf{u}}$, one for each monomial $\mathbf{z}^{\mathbf{u}}$ of degree $\leq d_i$. The total number of unknown coefficients equals $n = \sum_{i=1}^{m+1} \binom{d_i+m}{m}$. We write $\mathbb{Q}[\mathbf{x}, \mathbf{z}]$ for the resulting polynomial ring in $n + m$ variables. Inside this ring we consider the ideal

$$I \;=\; \langle\, f_1, f_2, \ldots, f_m, f_{m+1} \,\rangle \;\subset\; \mathbb{Q}[\mathbf{x}, \mathbf{z}].$$

We are interested in the ideal in $\mathbb{Q}[\mathbf{x}]$ obtained by eliminating the m variables $z_1, \ldots, z_m$:

Theorem 11. *The elimination ideal* $I \cap \mathbb{Q}[\mathbf{x}]$ *is principal. Its generator is an irreducible polynomial in the coefficients, denoted* $\mathrm{Res}(f_1, \ldots, f_{m+1})$ *and called the* resultant*. The degree of the resultant in the coefficients of* f_i *equals* $d_1 \cdots d_{i-1} d_{i+1} \cdots d_{m+1}$ *for* $i = 1, 2, \ldots, m+1$.

Proof. See [2, Chapter 3]. In that source, and many others, the f_i are taken to be homogeneous polynomials in $m+1$ variables. We here prefer the inhomogeneous formulation, which allows for a simpler formulation as an elimination ideal. The two versions are equivalent. $\square$

Example 12 (Determinants)**.** Let $d_1 = d_2 = \cdots = d_{m+1} = 1$. The $m+1$ polynomials f_i are affine-linear, and they can be expressed in the matrix-vector product

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \\ f_{m+1} \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} & x_{1,m+1} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} & x_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} & x_{m,m+1} \\ x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,m} & x_{m+1,m+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 1 \end{pmatrix}.$$

The resultant $\det(f_1, \ldots, f_{m+1})$ is the determinant of the coefficient matrix $(x_{i,j})$. This is a homogeneous polynomial of degree $m+1$ in $n = (m+1)^2$ unknowns having $(m+1)!$ terms.

Example 13 (Eliminating one variable from two quadratic polynomials)**.** Let $m = 1$ and $d_1 = d_2 = 2$ and abbreviate $z = z_1$. Then our system consists of two univariate polynomials

$$f_1 = x_{11}z^2 + x_{12}z + x_{13} \quad \text{and} \quad f_2 = x_{21}z^2 + x_{22}z + x_{23}.$$

The generator of the elimination ideal $\langle f_1, f_2 \rangle \cap \mathbb{Q}[\mathbf{x}]$ is the *Sylvester resultant*

$$\mathrm{Res}(f_1, f_2) \;=\; \det \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ 0 & x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} & 0 \\ 0 & x_{21} & x_{22} & x_{23} \end{pmatrix}. \tag{2}$$

This is a bi-homogeneous polynomial of bidegree $(d_1, d_2) = (2, 2)$. Its expansion has 7 terms.

The formula (2) generalizes to two polynomials in z of arbitrary degrees d_1, d_2. We set

$$\mathrm{Syl}_{d_1,d_2} \;=\; \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,d_1+1} & 0 & \cdots & 0 & 0 \\ 0 & x_{11} & x_{12} & \ddots & x_{1,d_1+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1,d_1+1} & 0 \\ 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1,d_1+1} \\ x_{21} & x_{22} & \cdots & x_{2,d_2+1} & 0 & \cdots & 0 & 0 \\ 0 & x_{21} & x_{22} & \ddots & x_{2,d_2+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2,d_2+1} & 0 \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2,d_2+1} \end{pmatrix}$$

This is the *Sylvester matrix* of format $(d_2{+}d_1) \times (d_2{+}d_1)$. For $d_1 = d_2 = 2$ it appears in (2).

Theorem 14. *The determinant of the matrix* Syl_{d_1,d_2} *is equal to the resultant* $\mathrm{Res}(f_1, f_2)$ *of*

$$f_1(z) \;=\; x_{11}z^{d_1} + \cdots + x_{1,d_1}z + x_{1,d_1+1} \quad \textit{and} \quad f_2(z) \;=\; x_{21}z^{d_2} + \cdots + x_{2,d_2}z + x_{2,d_2+1}.$$

Proof. We first note that the determinant $\det(\mathrm{Syl}_{d_1,d_2})$ is a non-zero polynomial. We see this by specializing $f_1 = z^{d_1}$ and $f_2 = 1$. Here the Sylvester matrix is the identity matrix.

Let Z denote the column vector with entries $z^{d_1+d_2-1}, z^{d_1+d_2-2}, \ldots, z^2, z, 1$, and let F denote the column vector with entries $z^{d_2-1}f_1, \ldots, zf_1, f_1,\ z^{d_1-1}f_2, \ldots, zf_2, f_2$. Both of these column vectors have length $d_1 + d_2$, and they are related by the Sylvester matrix:

$$\mathrm{Syl}_{d_1,d_2} \cdot Z \;=\; F.$$

If we multiply this equation on the left by the adjoint of the Sylvester matrix, then we obtain

$$\det(\mathrm{Syl}_{d_1,d_2}) \cdot Z \;=\; \mathrm{adj}(\mathrm{Syl}_{d_1,d_2}) \cdot F.$$

The last coordinate of the column vector Z equals 1. Hence the last coordinate in this equation shows that $\det(\mathrm{Syl}_{d_1,d_2})$ is a polynomial linear combination of the entries of F, and hence it lies in the ideal $\langle f_1, f_2\rangle$. The Sylvester determinant is a non-zero homogeneous polynomial of degree d_1+d_2 that lies in the ideal $\langle f_1, f_2\rangle \cap \mathbb{Q}[\mathbf{x}]$. We know from Theorem 11 that this ideal is principal, and its generator $\mathrm{Res}(f_1, f_2)$ also has degree $d_1 + d_2$. This implies that $\mathrm{Res}(f_1, f_2)$ is equal to $\det(\mathrm{Syl}_{d_1,d_2})$, up to a non-zero multiplicative constant. □

Example 15. Let $f_1(z), f_2(z)$ be univariate polynomials of degree d_1, d_2 in $\mathbb{Q}[z]$. This defines a map $f : \mathbb{C} \to \mathbb{C}^2$ whose closed image is an algebraic curve in the plane $\mathbb{C}^2$ with coordinates x_1, x_2. The implicit equation of this curve is the resultant $\mathrm{Res}_z\big(x_1 - f(z), x_2 - g(z)\big)$.

If $m \geq 2$ then the resultant $\mathrm{Res}(f_1, f_2, \ldots, f_{m+1})$ is more difficult to compute, and there does not always exists a formula as the determinant whose entries are linear expressions in the coefficients of $f_1, f_2, \ldots, f_{m+1}$. In some cases, however, such formulas are available in the literature. For instance, Sylvester already gave such a formula for $m = 2$ and $d_1 = d_2 = d_3$.

Exercises

1. Eliminate the variable z from the equations $x^3y^3z^3 - x - y - z = 1$ and $x^5 + y^5 + z^5 = 2$.

2. Prove: If an ideal I is prime then so are its elimination ideals, and same for radical.

3. Compute the determinants of the Sylvester matrices $\mathrm{Syl}_{1,5}$, $\mathrm{Syl}_{2,4}$ and $\mathrm{Syl}_{3,3}$. Each of them is a polynomial of degree 6 in 8 unknowns. Which of them has the most terms?

4. A plane curve has the parametrization $z \mapsto \big(f(z), g(z)\big)$ where f and g are polynomials of degree 10. At most how many terms do you expect the implicit equation to have?

5. Can you find an invertible 5×5-matrix that is skewsymmetric?

6. You are given all entries of a skewsymmetric 5×5 matrix $X = (x_{ij})$ except for x_{12} and x_{45}. Under which condition on the 8 visible entries can you complete with $\text{rank}(X) \leq 2$?

7. Let π be the linear map from $\mathbb{C}^3$ to $\mathbb{C}^2$ given by the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Given an algebraic curve V in $\mathbb{C}^3$, explain how one can compute the plane curve $\overline{\pi(V)} \subset \mathbb{C}^2$.

8. Consider the Fermat curve $V = \mathcal{V}(x^3 + y^3 + z^3)$ in the projective plane $\mathbb{P}^2$. Compute the ideal in 6 variables whose variety is the image of V under the *Veronese map*

$$\mathbb{P}^2 \to \mathbb{P}^5, \ (x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2).$$

9. Determine the prime ideal of relations among the 3×3-minors of a 3×6-matrix.

10. Let V_1 and V_2 be curves in $\mathbb{C}^3$ and $V_1 + V_2$ their pointwise sum. The Zariski closure $\overline{V_1 + V_2}$ is an algebraic variety in $\mathbb{C}^3$. Explain how one can compute its ideal $\mathcal{I}(V_1 + V_2)$.

11. Compute the hyperdeterminant of a $2 \times 2 \times 3$ tensor whose 12 entries are unknowns.

12. Apply the method in Example 15 to compute the implicit equation of the plane curve that has the parametrization $z \mapsto \big(2z^3 + 3z^2 + 5z + 7, \, 11z^3 + 13z^2 + 17z + 19\big)$.

13. Let $m = 2$, $d_1 = 1$, $d_2 = d_3 = 2$. The total number of coefficients is $n = 15 = 3+6+6$. Compute the resultant $\text{Res}(f_1, f_2, f_3)$ explicitly, as a polynomial in all 15 unknowns.

14. Which constraints hold for off-diagonal entries of a rank one 3×3-matrix?

15. Which constraints hold for off-diagonal entries of a nilpotent 3×3-matrix?

16. Which constraints hold for off-diagonal entries of an orthogonal 3×3-matrix?

17. Let $m = 2$ and $d_1 = d_2 = d_3 = 2$. Then $\text{Res}(f_1, f_2, f_3)$ is the resultant of three quadrics in the plane. This is a polynomial in $18 = 6+6+6$ variables of degree $12 = 4+4+4$. How many terms does it have? Find an explicit matrix formula for $\text{Res}(f_1, f_2, f_3)$.

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