

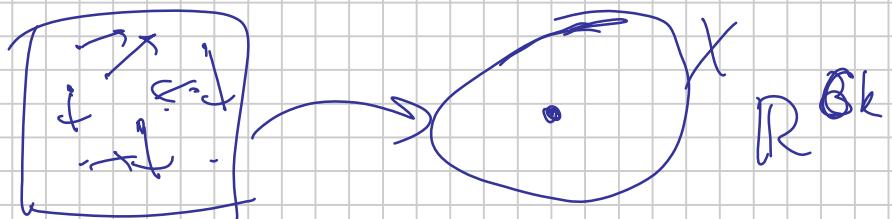
# Ring vorlesung

## Part 1: Ergodic Theory.

### 0. What is ergodic theory

Birth/origin: ~1880 Ludwig Boltzmann, statistical mechanics.

Model: Given: a region in  $\mathbb{R}^3$ ,  $b$  particles (ideal gas)  
 $\leadsto$   $6b$  coordinates (3 $b$  geom., 3 $b$  velocity)



State of the system = point in  $\mathbb{R}^{6b}$

Ergodic theory is theory of meas.-pres. transformations  
and studies the behaviour of orbits of points  
and sets.

Origin of "ergodic" not clear:  
ergodos = difficult  
ergon = work  
odos = path.

### 1. Measure-preserving systems: definition and examples

Def. 1.1 Let  $(X, \Sigma, \mu)$  be a prob. space (often  $(X, \mu)$ ),  
let  $T: X \rightarrow X$  measurable and  $\mu$ -pres.:  
 $\mu(T^{-1}(A)) = \mu(A) \quad \forall A \in \Sigma.$

$(X, \mu, T)$  is called a meas.-preserving dynam. system (MDS)

Rem. Images have larger measure:  $\mu(TA) \geq \mu(A)$

### Example 1.2 (Finite systems)

$X$  finite set,  $\mu$  rescaled counting  
measure,  $T: X \rightarrow X$  bij  
 $(X, \mu, T)$  is MDS

### Example 1.3 (Bernoulli shifts)

1) Two-sided shifts:

Let  $k \in \mathbb{N}$ ,

$T \leftarrow$  left shift:

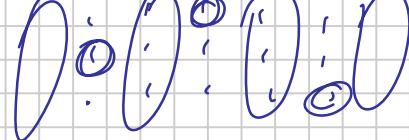
$$X := \{0, \dots, k-1\}^{\mathbb{Z}}$$
$$(x_j) \mapsto (x_{j+1})$$



$\Sigma$  := product  $\Sigma$ -algebra generated by cylinders.

$$U_{n, j_1, \dots, j_n, a_1, \dots, a_n} := \{X: X_{j_1} = a_1, \dots, X_{j_n} = a_n\}$$

$\Sigma$  is  $T$ -inv



Let  $p$  be a prob. measure on  $\{0, \dots, b-1\}$ ,  $p_j := p(\{j\})$

Def:  $\mu(U_{n, \dots}) = p_{a_1} \cdots p_{a_n}$

and extend to  $\Sigma$  (Carathéodory extension theorem)

$\mu$  is the product measure corr. to  $p$

We have:

$$\mu(T^{-1}(A)) = \mu(A) = \mu(TA) \quad \forall A \in \Sigma.$$

This MDS  $(X, \mu, T)$  is called the invertible (or two-sided) Bernoulli shift.

2) One-sided Shift:

$$X = \{0, \dots, k-1\}^{\mathbb{N}}$$

,  $T \leftarrow$ ,  $\Sigma$ ,  $\mu$  as before

Then:  $\mu(T^{-1}(A)) = \mu(A)$ .

This MDS is one-sided Bernoulli shift.

Example 1.4] (Group rotations)

Let  $G$  be compact group (comp. top. space, a group and  $\cdot: G \times G \rightarrow G$  and  $(\cdot)^{-1}: G \rightarrow G$  are continuous)

Then  $\exists!$  prob. measure  $\mu$  on the Borel  $\Sigma$ -algebra which is invariant under all left rotations:

$$\mu(gA) = \mu(A) \quad \forall g \in G \quad \forall A \in \Sigma.$$

$\mu$  is called the Haar measure on  $G$ .

(without proof).

## Properties of the Haar measure (no proof)

- $\mu$  is right invariant:

$$\mu(Ag) = \mu(A) \quad \forall g, A \in \Sigma$$

- $\mu$  is invariant w.r.t.  $g \mapsto g^{-1}$ , i.e.,

$$\mu(A^{-1}) = \mu(A) \quad \forall A \in \Sigma.$$

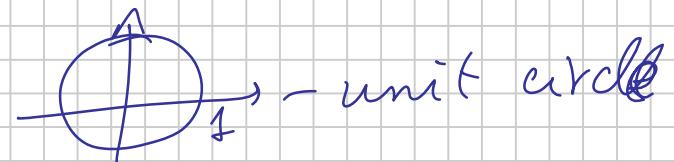
- $\text{supp } \mu = G$ , i.e.,  $\forall U \subset G$  open  $\mu(U) > 0$ .

### Examples

- $G$  finite discrete group,  $\mu(A) = \frac{|A|}{|G|}$

- $([0, 1), + \bmod 1)$   
Leb = Haar

- $T := \{z \in \mathbb{C} : |z| = 1\}$



- unit circle

$$\mu = \frac{\text{seb}}{2\pi} - \text{haar}$$

Attention: For a fixed  $g$  there can be many  $g$ -inv. measures.

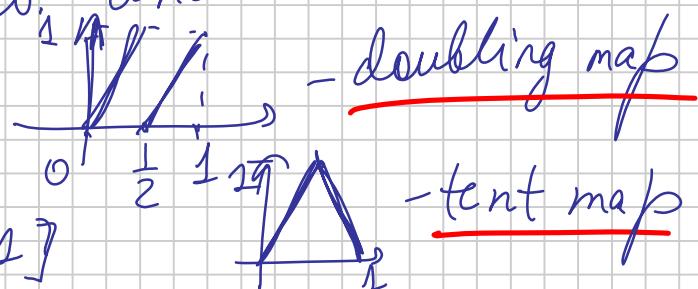
Homework (HW): 1)  $\mathbb{T}$ , let  $g$  be rational ( $\exists n: g^n = 1$ )  
Describe all  $g$ -inv. prob. measures on  $\mathbb{T}$

2)  $\mathbb{T}$ , let  $g$  be irrat. Show that the above  
 $\mu = \frac{\text{seb}}{2\pi}$  is the only one  $g$ -inv. prob. measure  
 on  $\mathbb{T}$  (Hint: Show first that  $\{g^n, n \in \mathbb{N}\}$   
 is dense in  $\mathbb{T}$ )

3) Show that  $[0, 1]$  with  $\text{seb}_1$  and

a)  $Tx = 2x \bmod 1$

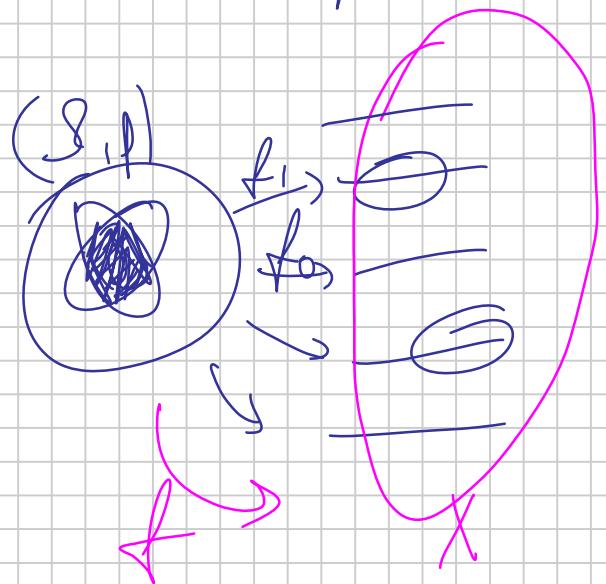
b)  $Tx = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2-2x, & x \in [\frac{1}{2}, 1] \end{cases}$



are MDS

4) stationary stochastic processes  $\leftrightarrow$  shift invariant measures on  $(\mathbb{R}^{\mathbb{Z}})$

Let  $(\Omega, \mathcal{P})$  prob-space,  $\dots, f_{-1}, f_0, f_1, f_2, \dots : \Omega \rightarrow \mathbb{R}$  meas.  
Assume: the process is stationary, (i.e., random variables)



$$P\left(w \in \Omega : \begin{array}{c} f_{n_1}(w) \in B_1 \\ \vdots \\ f_{n_m}(w) \in B_m \end{array}\right) = P\left(w \in \Omega : \begin{array}{c} f_{n_1+k}(w) \in B_1 \\ \vdots \\ f_{n_m+k}(w) \in B_m \end{array}\right)$$

$\forall k, n_1, \dots, n_m \in \mathbb{Z} \quad \forall B_1, \dots, B_m \subset \mathbb{R}$

Def.  $f : \Omega \rightarrow X$  with  $X := \mathbb{R}^{\mathbb{Z}}$  meas.

$$f(w) = (\dots, f_{-1}(w), f_0(w), f_1(w), \dots)$$

For  $A \subset X$  meas.: def.

$$\mu(A) := p(f^{-1}(A)).$$

Show that  $(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow)$  is an MDS

so: stationary process  $\xrightarrow{\text{left shift: } T(x_j) = (x_{j+1})}$   $\xrightarrow{\text{shift m. measure on } \mathbb{R}^{\mathbb{Z}}}$

Converse: If  $(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow)$  is an MDS, then  $\exists (\mathfrak{L}, p, (f_i))$   
which induces  $(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow)$

### Ex 1.5 [Product systems]

Let  $(X, \mu, T)$ ,  $(Y, \nu, S)$  be MDS. The product system  
 $(X \times Y, \mu \times \nu, T \times S)$  is again an MDS, where!

- $(T \times S)(x, y) = (Tx, Sy)$

- $\Sigma$ -algebra is generated by  $\bigcup_{A \in \Sigma} A \times B$
- $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$ .

Concrete ex.:  $\mathbb{T}^d$ ,  $T(x_1, \dots, x_d) = (a_1 x_1, \dots, a_d x_d)$   
 (also: rotation on a comp. group). for fixed  $a_1, \dots, a_d \in \mathbb{T}$

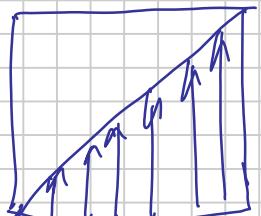
[Ex. 1.6] (skew shift)

Consider  $\mathbb{T}^2$ ,  $S(x, y) := (ax, xy)$  for a fixed  $a \in \mathbb{T}$ .

HW

$(\mathbb{T}^2, \text{Leb}, S)$  is an MDS

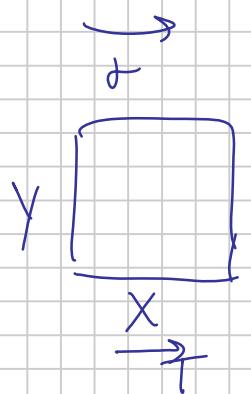
An additive version:  $([0, 1]^2, \text{Leb}, S)$



HW

Compute  $S^n(x, y)$  for all  $n \in \mathbb{N}$ ,  $(x, y)$ .

$$S(x, y) := (x + a, x + ay)$$



More generally, let  $(X, \mu, \tau)$  be MDS,  $(Y, \nu)$  prob. space,  
 $\varphi: X \times Y \rightarrow Y$  meas. transf. s.t.

for a.e.  $x$ ,  $\varphi(x, \cdot): Y \rightarrow Y$  is  $\nu$ -pres.

$$(\text{e.g., } \nu(A) = \nu(\{y \in Y : \varphi(x, y) \in A\})$$

Def.  $S: X \times Y \rightarrow X \times Y$  by  $S(x, y) := (\tau x, \varphi(x, y))$

Then  $(X \times Y, \mu \times \nu, S)$  is an MDS HW

if  $Y = G$  <sup>is</sup> comp. group with  $\nu = \text{Haar}$ ,  $\varphi(x, g) = \psi(x) \cdot g$

for  $\psi: X \rightarrow G$  meas., then this MDS is called

a group extension.

In our skew shift above:  $\tau$ -extension of the rotation  $(T, \text{Leb}, \alpha)$ .

Lemma 1.7] Let  $(X, \mu)$  be prob. space,  $T: X \rightarrow X$  meas. Then:

$T$  is  $\mu$ -pres. ( $\Leftrightarrow$ )

(\*)

$$\int f \circ T d\mu = \int f d\mu \quad \forall f \in L^\infty(X, \mu)$$

in this case, (\*) holds for  $\forall f \in L^1(X, \mu)$ .

Proof Observe:

$$\boxed{\mathbb{1}_A(Tx) = \mathbb{1}_{T^{-1}(A)}(x)}$$

so: (\*) for  $\mathbb{1}_A$  ( $\Rightarrow \mu(T^{-1}(A)) = \mu(A)$ )

Rest: 

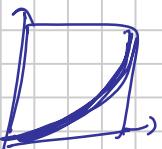


2. Topological dynam. systems  
and invariant measures.

\* A topological dynamical system (TDS) is a pair  $(X, T)$ , where

- $X$  compact metric space
- $T: X \rightarrow X$  continuous

[Ex]:  $[0,1]$ ,  $Tx = x^2$



Question:  $\exists \mu$  Borel prob. measure on  $X$  which is  $T$ -inv.,  
(e.g., s.t.  $(X, \mu, T)$  is an MDS?)

In the above ex.:  $\delta_0, \delta_1$  are  $T$ -inv.  $O \in A \Leftrightarrow O \in T^{-1}(A)$   
 $1 \in A \Leftrightarrow \dots$

Rem. 2.1  $\forall U \subset X$  open  $\exists (f_n) \subset C(X)$  with:  
 $O \subseteq f_n^{-1}U$

HW Show it directly (without Urysohn) using distance  $d$ .

Lemma 2.2

For a TDS  $(X, T)$  and a Borel prob. measure  $\mu$  on  $X$ ,

$T$  is  $\mu$ -pres.  $\Leftrightarrow$

$$\int f d\mu = \int f \circ T d\mu \quad \forall f \in C(X).$$

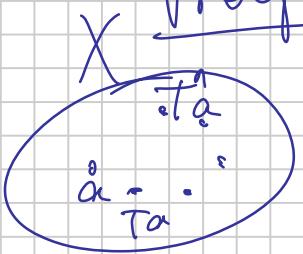
Proof: MW

Thm 2.3

Krylov - Bogolyubov

For  $\forall$  TDS  $(X, T) \exists$   $T$ -inv. Borel prob. measure  $\mu$  on  $\overline{T}$ .

Proof Take  $a \in X$  arb. and consider:



$$\mu_n := \frac{\delta_a + \delta_{Ta} + \dots + \delta_{T^n a}}{n+1}$$

$\forall \mu_n$  is a Borel prob. measure on  $X$  and :

$$\left| \int_X (f(Tx) - f(x)) d\mu_n \right| = \left| \frac{1}{n+1} \left[ f(Ta) + \dots + f(T^{n+1}a) - f(a) - \dots - f(T^n a) \right] \right| = \frac{\left| f(T^{n+1}a) - f(a) \right|}{n+1} \leq \frac{2 \|f\|_\infty}{n+1} \xrightarrow{n \rightarrow \infty} 0$$

Let  $\mu$  be a limit point of  $\{\mu_n\}_{n=1}^\infty$  w.r.t. weak\* topology

(recall:  $(C(X))' = \mathcal{M}(X)$  with  $\mathcal{M}(X)$  <sup>R</sup>Borel measures on  $X$ )  
+ Banach-Alaoglu theorem: unit ball of  $C(X)'$  is weak\*-comp.)

Then:

- $\mu$  is a Borel prob. measure /  $\mu(X) = \int_X 1 d\mu = 1$

- $\mu$  is  $T$ -inv:  $\int f(Tx) d\mu = \int f d\mu = 0 \quad \forall f \in C(X)$

Lemma 2.2 - finished

Remark 1) So: TDS  $\leadsto$  MDS (in general not unique: ex:  $T = \text{id}$ )

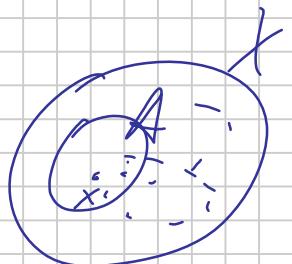


2) if  $\mu$  is unique, then  $(X, T)$  is called uniquely ergodic

### 3. Recurrence

[Def. 3.1] Let  $(X, \mu, T)$  be MDS,  $A \subset X$  meas.

$A$  is called:



1) recurrent, if for a.e.  $x \in A$   $\exists n : T^n x \in A$

2)  $\infty$ -recurrent, if for a.e.  $x \in A$   $\exists (n_j) \subset \mathbb{N} :$   
 $T^{n_j} x \in A \quad \forall j$ .

Rem. 1)  $A$  is recurrent ( $\Rightarrow$ )

$$\mu(A \setminus \bigcup_{n=1}^{\infty} T^{-n}(A)) = 0$$

$\Rightarrow$   $A$  is  $\infty$ -recurrent

$$\mu(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} T^{-n}(A)) = 0$$

2)  $\mu(A)=0 \Rightarrow \infty$ -recurrent (trivial case).

Lemma 3.2: For an MDS  $(X, \mu, T)$  TFAE:

(i)  $\forall A \in \Sigma_X$  is recurrent

(ii)  $\forall A \in \Sigma_X$  is  $\infty$ -recurrent

(iii)  $\forall A \in \Sigma_X$  with  $\mu(A)>0$   $\exists n: \mu(A \cap T^{-n}A) > 0$

Proof

$(ii) \Rightarrow (i)$  ✓

$(i) \Rightarrow (ii)$  (i) implies:

$A \subset (T^{-1}(A) \cup T^{-2}(A) \cup \dots) \cup N$  null set

Then:  $T^{-1}(A) \subset (T^{-2}(A) \cup T^{-3}(A) \cup \dots) \cup T^{-1}(N)$

$T^{-2}(A) \subset (T^{-3}(A) \cup \dots) \cup T^{-2}(N)$

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so we have:  $A \subset \left( T^{-k}(A) \cup \dots \right) \cup \underbrace{\left( N \cup T^{-1}(N) \cup \dots \cup T^{-(k-1)}(N) \right)}_{\text{null set}}$

for  $\forall k$ .

$(i) \Rightarrow (iii)$  Assume:  $\mu(A \cap T^{-n} A) = 0 \quad \forall n$ .

To show:  $\mu(A) = 0$ .

By (i):

$$A = [A \cap \left( \bigcup_{n=1}^{\infty} T^{-n} A \right)] \cup N$$

$\text{null set}$

$$\bigcup_{n=1}^{\infty} A \cap T^{-n} A \text{ - null set}$$

$(iii) \Rightarrow (i)$  For  $A \in \Sigma_b$  consider:

$$B := A \setminus \bigcup_{k=1}^{\infty} T^{-k} A = A \cap \left( \bigcap_{k=1}^{\infty} T^{-k}(X \setminus A) \right)$$

To show:  $\mu(B) = 0$ .

$$\text{For } \forall n \in \mathbb{N} \quad B \cap T^{-n}B \subset A \cap T^{-n}A \cap \left( \bigcap_{k=1}^{\infty} T^{-k}(X \setminus A) \right) = \emptyset$$

□

$$(iii) \Rightarrow \mu(B) = 0.$$

Thm 3.3 (Poincaré)

For an MDS  $(X, \mu, T)$ , every  $A \in \Sigma_X$  is  $\sigma$ -recurrent.

Proof Take  $A \in \Sigma_X$  with  $\mu(A) > 0$  ( $\mu(A) = 0 \Rightarrow \sigma\text{-recur.}$ )

To show (Lemma 3.2):  $\mu(A \cap T^{-n}(A)) > 0$  for some  $n$ .

Assume not:  $\forall n \quad \mu(A \cap T^{-n}(A)) = 0$

Then:

$$\mu(T^{-(n+k)}(A) \cap T^{-k}(A)) = 0 \quad \forall n, k.$$

$T^{(n+k)}(T^{-n}A \cap A)$

The sets

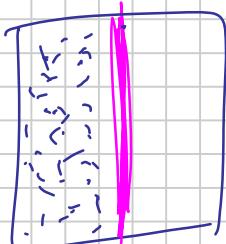
$A, T^{-1}(A), T^{-2}(A), \dots$  are disjoint (up to null sets) and have measure  $\mu(A) > 0$ , so:

$$1 = \mu(X) \geq \mu\left(\bigcup_{n=1}^{\infty} T^{-n} A\right) = \sum \mu(A) = \infty \quad \text{[REDACTED]}$$

Rem. Poincaré's thm has counterintuitive conseq.

Boltzmann's model:

The state space  $X \subset [0, 1]^{3n} \times \mathbb{R}^{3n}$



Consider  $A := \{x \in X : x_1 < \frac{1}{2}\}$  - a wall in the middle, remove it, time goes.

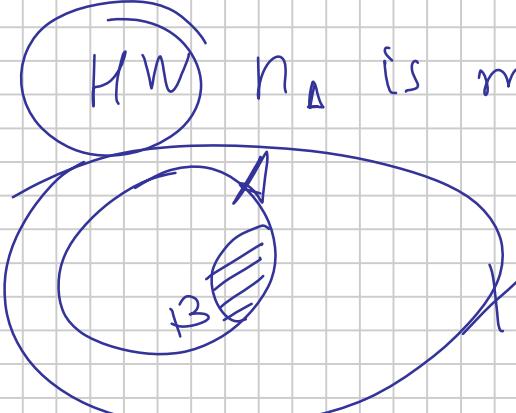
Poincaré: for a.e. initial state,  $\exists n : T^n x \in A$

Question: When do points return to  $A$ ? (How long should one wait?)

Define the return time fct.: (assume  $\mu(A) > 0$ )

$$n_A : A \rightarrow \mathbb{N} \cup \{\infty\}$$

$$n_A(x) := \text{the smallest } n : T^n x \in A$$

  $n_A$  is measurable w.r.t. the induced  $\sigma$ -algebra / measure:

$$\Sigma_A := \{B \subset A : B \in \Sigma_X\} = A \cap \Sigma_X \subset \Sigma_X$$

$$\mu_A(B) = \frac{\mu(B)}{\mu(A)}$$

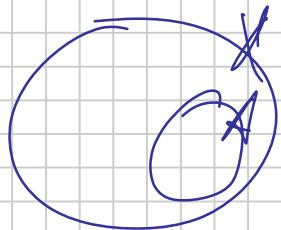
One can show (no proof here).

Let  $(X, \mu, T)$  be MDS,  $A \subset X : \mu(A) > 0$ . Then:

$$\int_A n_A d\mu_A = \frac{\mu(\bigcup_{n=0}^{\infty} T^{-n}(A))}{\mu(A)}$$

points from X visiting A at least once.

expected value of  $n_A$



Ex

Boltzman:  $A = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$  -  $\mu(A) = \frac{1}{2^k}$

The expected value of  $n_A$  is  $\sim 2^k$  

$$\bigcup_{n=0}^{\infty} T^{-n}(A) = X \text{ a.e.}$$

When is it true?

#### 4. Ergodicity

Def 4.1

Let  $(X, \mu, T)$  be MDS,  $A \in \Sigma$  is called invariant

if

$T^{-1}(A) \subset A$  up to a null set, i.e.,

$$\mu(T^{-1}(A) \setminus A) = 0.$$

Lemma 4.2

TFAE:

(i)  $A$  is invariant

(ii)  $A = T^{-1}(A)$  up to a null set

(iii)  $X \setminus A$  is invariant.

Proof  $(i) \Rightarrow (ii)$  follows from  $\mu(A) = \mu(T^{-1}(A))$   
 $(ii) \Rightarrow (iii)$  By (ii)

$$T^{-1}(X \setminus A) = X \setminus T^{-1}(A) = X \setminus A$$

$(iii) \Rightarrow (i)$  symmetry  $A \leftrightarrow X \setminus A$ .



Rem. i) (ii)  $\Rightarrow T A = A$  a.e.

2)  $A$  inv.,  $\mu(A) \in (0, 1)$   $\Rightarrow$  2 subsystems  
 $(A, \Sigma_A, \mu_A, T)$  and  $(X \setminus A, \Sigma_{X \setminus A}, \mu_{X \setminus A})$

Def. 4.3 An MDS  $(X, \mu, T)$  is called ergodic if  $\forall A \in \Sigma$ :  
 $A$  is invariant  $\Rightarrow \mu(A) \in \{0, 1\}$

Rmk. 1) Erg.-systems are indecomposable.

2)  $\exists$  MDS without erg. subsystem : [Ex]:  $X = \{0, 1\}$

$$\begin{aligned}\mu &= \text{LeB} \\ T &= \text{id}\end{aligned}$$

[Prop. 4.4] For an MDS  $(X, \mu, T)$  TFAE:

(i)  $(X, \mu, T)$  is ergodic

(ii)  $\forall A$  with  $\mu(A) > 0$

$$\bigcap_{n=0}^{\infty} \bigcup_{k \geq n} T^{-k}(A) = X \quad \text{a.e.}$$

(iii)  $\forall A$  with  $\mu(A) > 0$

$$\bigcap_{n=0}^{\infty} T^{-n}(A) = X \quad \text{a.e.}$$

(iv)  $\forall A, B \subset X$  with  $\mu(A) > 0, \mu(B) > 0 \quad \exists n \geq 1:$

$$\mu(T^{-n}(A) \cap B) > 0$$

Proof  $\left( \text{i} \Rightarrow \text{ii} \right)$  Let  $A$  with  $\mu(A) > 0$ . Define

$$B_n := \bigcup_{k \geq n} T^{-k}(A).$$

We have:  $T^{-1}(B_n) = \bigcup_{k \geq n+1} T^{-k}(A) \subset B_n$ , i.e.,  $B_n$  is invariant.

$$\text{But } \mu(B_n) \stackrel{\infty}{\geq} \mu(A) > 0, \text{ so: } \mu(B_n) = 1 \quad \forall n$$

$$\Rightarrow \mu\left(\bigcap_{n=0}^{\infty} B_n\right) = 1$$

$\left( \text{iii} \Rightarrow \text{iv} \right)$  clear?

$\left( \text{iv} \Rightarrow \text{v} \right)$  Let  $\mu(A), \mu(B) > 0$ , assume  $\mu(T^{-n}(A) \cap B) = 0 \quad \forall n \geq 1$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} (T^{-n}(A) \cap B)\right) = 0. \text{ But:}$$

$$\bigcup_{n=1}^{\infty} (T^{-n}(A) \cap B) = \left[ \bigcup_{n=1}^{\infty} T^{-n}(A) \right] \cap B = \left[ T^{-1}\left(\bigcup_{n=0}^{\infty} T^{-n}(A)\right) \right] \cap B$$

$$\Rightarrow \mu(B) = 0$$

$$\underbrace{\mu = 1}_{\mu = 1 \text{ by iii}}$$

(iv)  $\Rightarrow$  (i) Let  $A$  be invariant:  $T^{-1}(A) \subset A$  a.r.

$\Rightarrow T^{-n}(A) \subset A$  a.r.

For  $B := X \setminus A$  we have:

$$B \cap T^{-n}(A) \subset B \cap [A \cup N] = [B \cap A] \cup [B \cap N]$$

$\xrightarrow{\text{null set}}$

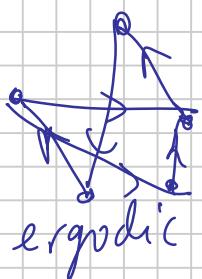
$\xrightarrow{\text{null set}}$

By (iv),  $\mu(B) = 0$  or  $\mu(A) = 1$

[Ex] (Finite systems)

$X$  finite,  $\mu$  rescaled counting measure,  $T: X \rightarrow X$   $\delta_{ij}$   
(then  $\mu$ -pres.)

Then  $T$  is erg. ( $\Leftrightarrow$ )  $\forall$  point reaches  $\forall$  other point,  
i.e.,  $T$  is cyclic.



not erg.

(KW)

Show that  $T$  erg.  $\not\Rightarrow T^2$  erg. in general

What is about  $\Leftarrow$ ?

[Ex]

Bernoulli shifts

Bernoulli shifts satisfy a much stronger property:

$$(*) \quad \mu(T^{-n}(A) \cap B) \xrightarrow{n \rightarrow \infty} \mu(A) \cdot \mu(B) \quad \forall A, B \in \mathcal{E}$$

In particular, (iv) in Prop. 4.4 is satisfied  $\Rightarrow$  erg.

(KW)

: Proof : (Hint: step 1:  $A, B$  cylinder sets := for large  $n$ 's)  
step 2: finite unions  
step 3: general sets.

Rem. Property (\*) is called mixing (or strong mixing)

[Thm 4.5] (Kac) Let  $(X, \mu, T)$  be an ergodic MDS and

$\forall X$  with  $\mu(A) > 0$ . Then the expected return time to  $A$  satisfies

$$\int_A n_A d\mu_A = \frac{1}{\mu(A)}$$

Proof: By Prop. 4.4,  $\bigcup_{n=0}^{\infty} T^{-n}(A) = A$  a.s. + Prop. 4.3 "■"

Ex (Recurrence in random literature)

Nietzsche's book "Thus spoke Zarathustra" has  $\sim 680,000$  characters. Assume: Snoopy types randomly on a typewriter with 90 symbols.

Claim: He types Nietzsche's book  $\infty$ -often

HW 1) Proof using the bernoulli shift  $B(\underbrace{\frac{1}{90}, \dots, \frac{1}{90}}_{90 \text{ times}})$  and an appropriate cylinder set  $A$ .  
Use ergodicity and Prop. 4.4

2) Compute the expected return time.

(Reformulation of ergodicity using fcts)

[Def 4.6] For  $p \in (1, \infty)$  consider:  $E := L^p(X, \mu)$  and

operator  $(Tf)(x) := f(Tx)$

transformation on  $X$

$T: E \rightarrow E$  is called the Koopman operator corresp. to

Properties of  $T$  on  $E$  (HW)

$(X, \mu, T)$

•  $T$  lin. and isometric (i.e.,  $\|Tf\|_p = \|f\|_p$ )

•  $T$  is unitary if  $p=2$  and if the transf.  $T$  is invertible  
(i.e.,  $\exists S: X \rightarrow X$   $\mu$ -pres. with  
 $S \circ T = id = T \circ S$  a.e.)

•  $T1 = 1$  and  $T1_A = 1_{T^{-1}(A)}$ .

- $T$  is positive :  $f \geq 0 \Rightarrow Tf \geq 0$
- $T$  is multiplicativ and conjugation invariant :  

$$T(f \circ g) = Tf \circ Tg, T(\overline{f}) = \overline{Tf}$$
  
if the product is in  $E(1)$
- $|Tf| = T|f| \quad \forall f$  -  $T$  is a lattice homomorphism

Rem. If operator on  $L^p(X, \mu)$  with these properties ( $\geq 0$ , mult., etc.)  
 is a Koopman operator.

# Ergodic theory: continuation

Notiztitel

25.04.2017

Prop. 4.7

Let  $(X, \mu, T)$  be MDS,  $p \in [1, \infty)$  and  $T$  be the Koopman oper. on  $E := L^p(X, \mu)$ . Consider

$$\text{Fix } T := \{f \in E : Tf = f\}$$

TFAE:

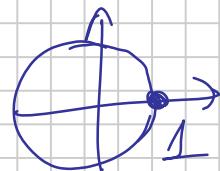
(i)  $(X, \mu, T)$  is ergodic

(ii)  $\dim \text{Fix } T = 1$ , i.e., 1 is a simple eigenvalue

with

$$\text{Fix } T = \mathbb{C} \cdot 1$$

in this case,  $\forall$  eigenfct has constant absolute value:  
 $T\lambda = \lambda f \Rightarrow |f| = \text{const.}$



Proof Last assertion ("in this case"):

$$\begin{cases} Tf = \lambda f \Rightarrow |\lambda| = 1 \Rightarrow |Tf| = |f| \Rightarrow f \in \text{Fix } T \\ f \neq 0 \end{cases}$$

T is isom.

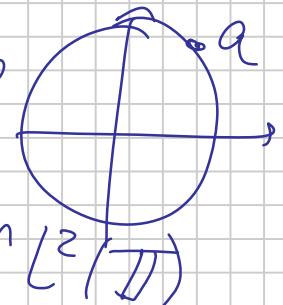
$T \not\cong H$

$$(ii) \Rightarrow |f| = c \cdot 1$$

Rest (i)  $\Leftrightarrow$  (ii) : HW (Hint for (i)  $\Rightarrow$  (ii)) : for  $f \in \text{Fix } T$  real valued  
 consider:  $A_c := \{x : f(x) \leq c\} \dots$  ■

### Ex (Rotations on $\mathbb{T}$ )

$(\mathbb{T}, a)$  is erg.  $\Leftrightarrow a$  is irrat., i.e.,  $a^n \neq 1 \quad \forall n \neq 0$



Proof HW (Hint: for  $\Leftarrow$  take  $f \in \text{Fix } T$  for  $T$  on  $L^2(\mathbb{T})$   
 and write  $f = \sum_{n=-\infty}^{\infty} c_n X_n$  for  
 $X_n(z) := z^n, \quad n \in \mathbb{Z}$  (ONB-orthon. basis in  $L^2(\mathbb{T})$ )

show that  $c_n = 0$  for  $\forall n \neq 0$  )

Rem. Analogously:

$(\mathbb{P}^d, a)$  is erg. ( $\Rightarrow (a_1, \dots, a_d)$  are  $\mathbb{Z}$ -indep.)

$$a_1^{n_1} \cdot \dots \cdot a_d^{n_d} = 1 \Rightarrow n_1 = \dots = n_d = 0$$

### 3. The mean ergodic theorem

Recall: Boltzmann's ergodic hypothesis:  $\frac{1}{N} \sum_{n=1}^N (f^n f)(x) \rightarrow \int f d\mu$

("time mean = space mean")

Def 5.1 Let  $(a_n) \subset \mathbb{C}$  bdd (more gener.:  $(a_n) \subset \text{top. space}$ ,  $a_n = o(n)$ ).  
We say:  $a_n$  conv. to  $a$  in the Cesàro sense if

$$\frac{a_1 + \dots + a_N}{N} \rightarrow a.$$

Write : C-lim  $a_n = a$  or  $a_n \xrightarrow{\text{Cesàro}} a$ .

[Ex]

i)  $0, 1, 0, 1, 0, 1, \dots \xrightarrow{\text{Cesàro}} \frac{1}{2}$

$0, 1, 1, 0, 1, 1, \dots \xrightarrow{\text{Cesàro}} \frac{2}{3}$

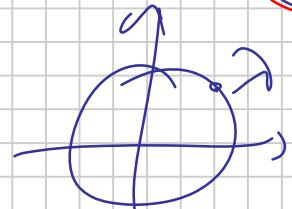
$(a_n)$  periodic with period  $d \Rightarrow a_n \xrightarrow{\text{Cesàro}} \frac{a_1 + \dots + a_d}{d}$

HW

2) For  $\lambda \in \mathbb{T}$

$$\text{C-lim } \lambda^n = \begin{cases} 0, & \lambda \neq 1 \\ 1, & \lambda = 1 \end{cases}$$

$$\left( \frac{1}{N} \sum_{n=1}^N \lambda^n = \frac{1}{N} \cdot \lambda \cdot \frac{1 - \lambda^N}{1 - \lambda} \xrightarrow[N \rightarrow \infty]{} 0 \text{ if } \lambda \neq 1 \right)$$



Properties of the Cesàro limit

1)  $a_n \rightarrow a \Rightarrow a_n \xrightarrow{\text{Cesàro}} a$

HW

2)  $\exists$  odd  $(a_n)$ :  $(a_n)$  diverges in the Cesàro sense

$$1, 0, \dots, 0, 1, \dots \quad \text{---} \quad 1, 0, \dots$$

3)  $C\text{-}\lim a_{n+1} = C\text{-}\lim a_n$  (shift invariant)

$$\left| \frac{a_2 + \dots + a_n}{n} - \frac{a_1 + \dots + a_n}{n} \right| = \frac{|a_{n+1} - a_1|}{n} \rightarrow 0$$

4) Attention:  $C\text{-}\lim$  is not multiplicative:

$$(a_n): 0, 1, 0, 1, 0, 1, \dots \xrightarrow{\text{Cesàro}} \frac{1}{2}$$

$$(b_n): 1, 0, 1, 0, 1, 0, \dots \xrightarrow{\text{Cesàro}} \frac{1}{2}$$

$$(a_n b_n): 0, 0, \dots \xrightarrow{\text{Cesàro}} 0 \neq \frac{1}{4}$$

Rem. Analog. one defines Cesàro conv. in a Banach space  
for the strong/weak conv. Also for operators:

strong/weak operator conv.

Return to erg. theory: For  $(X, \mu, T)$  MDS, def.

$$S_N f := \frac{1}{N} \sum_{n=1}^N T^n f \quad \text{for } f \in L^2(X, \mu)$$

let  $H := L^2(X, \mu)$

Thm 5.2 (von Neumann 1933 : mean erg. thm)

For an MDS  $(X, \mu, T)$  and  $f \in L^2(X, \mu)$ ,

$$\frac{1}{N} \sum_{n=1}^N T^n f \xrightarrow{\quad} P_f \quad \text{in } L^2,$$

where  $P$  is the orthog. proj. onto  $\text{Fix } T \subset L^2(X, \mu)$ .

Rem. „Mean“ in the name refers to  $L^2$  (not to  $\frac{1}{N} \sum_{n=1}^N$ )

Proof

Step 1: orthogonal decompr.

We first show:

$$H = \text{Fix } T \oplus \overline{\text{Rg}(\mathbf{I}-T)}$$

1)  $\text{Fix } T \perp \overline{\text{Rg}(\mathbf{I}-T)}$ :

Let  $f \in \text{Fix } T$ :  $Tf = f$ , and  $g \in H$ . Then:

$$\begin{aligned}\langle f, g - Tg \rangle &= \langle f, g \rangle - \underbrace{\langle Tf, Tg \rangle}_{= \langle f, g \rangle} = 0 \\ &\quad \text{since } T \text{ is isom.}\end{aligned}$$

2) Let  $f \perp \overline{\text{Rg}(\mathbf{I}-T)}$ . To show:  $f \in \text{Fix } T$ .

In fact.,  $0 = \langle f, f - Tf \rangle = \|f\|^2 - \langle f, Tf \rangle$ , i.e.,

$$\langle f, Tf \rangle = \|f\| \cdot \|Tf\|$$

" = " in the Cauchy-Schwarz ineq. implies:  $Tf = c \cdot f$   
so:  $\|f\|^2 = \langle f, Tf \rangle = \overline{c} \|f\|^2 \Rightarrow c = 1$ !

## Step 2 : Convergence

By Step 1 :  $f = \underbrace{f_1}_{\text{Fix } T} + \underbrace{f_2}_{\text{on } \overline{\text{Rg}(I-T)}}$

To show:  $S_N f \rightarrow f_1$ , i.e.,  $\begin{cases} S_N \rightarrow I \text{ on Fix } T \\ S_N \rightarrow 0 \text{ on } \overline{\text{Rg}(I-T)} \end{cases}$

Let  $f = g - Tg$ :

$$S_N f = \frac{1}{N} \sum_{n=1}^N (T^n g - T^{n+1} g) = \frac{1}{N} (Tg - T^{N+1} g)$$

$$\|S_N f\|_2 \leq \frac{2 \|g\|_2}{N} \xrightarrow[N \rightarrow \infty]{} 0$$

telescopic sum

Conv. on a dense subset + uniform bound ( $\|S_N\| \leq 1$ )

$\Rightarrow$  conv. on  $\overline{\text{Rg}(I-T)}$ .



- Rem.
- 1) The projection  $P$  is called the mean ergodic projection.
  - 2) The thm (not the proof!) holds for a much larger class of operators.

Prop. 5.3 (Characterisation of ergodicity)

For an MDS  $(X, \mu, T)$  with Koopman op.  $T$  on  $L^2(X, \mu)$  and  $m$ -erg. proj.  $P$  TFAE:

(i)  $(X, \mu, T)$  is erg.

(ii)  $\dim \text{Fix } T = 1$

(iii)  $\frac{1}{N} \sum_{n=1}^N T^n f \xrightarrow{\text{in } L^2} \int f d\mu \cdot \underline{1}$ , i.e.,

$$Pf = \int f d\mu \cdot \underline{1}$$

$$(iv) \quad \frac{1}{N} \sum_{n=1}^N \langle T^n f, g \rangle \rightarrow \int f d\mu, \int g d\mu \quad \forall f, g \in L^2$$

$$(v) \quad \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}B) \rightarrow \mu(A) \mu(B) \quad \forall A, B \in \Sigma$$

$$(vi) \quad \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A) \rightarrow \mu(A)^2 \quad \forall A \in \Sigma.$$

proof

$$(i) \Rightarrow (ii)$$

prof. 4.7 above

$$(ii) \Rightarrow (iii)$$

We know:  $Pf \in \text{Fix } T = C \cdot \mathbb{1} \Rightarrow Pf = c \cdot \mathbb{1}$

By

$$\int f d\mu = \int T^n f d\mu = \int Pf d\mu = c$$

clear

$$(iii) \Rightarrow (iv)$$

Take  $f := \mathbb{1}_B, g := \mathbb{1}_A$

$$(iv) \Rightarrow (v)$$

trivial

$$(v) \Rightarrow (vi)$$

let  $\star$  be  $T$ -inv.

(vi)

To show:  $\mu(A) \in \{0, 1\}$

$$(vi) \Rightarrow (i)$$

Then  $\frac{1}{n} \sum_{i=1}^n \mu(A \cap T^{-i}A) \xrightarrow{\text{m(A)}} \mu(A)^2$  :  $\mu(A) = \mu(A)^2$  

## 6. The pointwise ergodic theorem

Thm. 6.1 (Birkhoff 1933, the pointwise erg. thm)

For an MDS  $(X, \mu, T)$  and  $f \in L^1(X, \mu)$ ,

$$(\text{Ave}) \quad \frac{1}{n} \sum_{n=1}^N (T^n f)(x)$$

conv. a. e. (to  $(Pf)(x)$ ) .

Corollary 6.2 An MDS  $(X, \mu, T)$  is ergodic ( $\Rightarrow$ )  
 $\forall f \in L^1$  , time mean = space mean  $^u$ .

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (T^n f)(x) = \int f d\mu$$

for  $\forall x \in X$   
Proof (using Birkhoff)  $\square$

Observation: On the dense subspace

$$D := \text{Fix } T^\perp \oplus (I - T)(L^\infty(X, \mu)) \subset L^\perp$$

(Ave) conv. a. l. (even in  $L^\infty$ ):

- $Tf = f \Rightarrow \text{Ave} = f(x) \rightarrow f(x)$  a. l.

- $f = g - Tg$ ;  $g \in L^\infty$ , then

$$\begin{aligned} (\text{Ave}) &= \frac{1}{N} \sum_{n=1}^N (T^n g - T^{n+1} g) = \frac{Tg - T^{N+1} g}{N} \\ \| \cdot \|_\infty &\leq \frac{2 \| g \|_\infty}{N} \rightarrow 0 \end{aligned}$$

Problem: How to come from  $\mathcal{D}$  to  $\mathcal{D}^{1+1} = \mathbb{L}^1$ ?

Density argument?

Claim: A.e. conv. of fcts is not induced by any topology

(HW) Hint: Show first: in a top. space,  
 $a_n \rightarrow a \iff \forall$  subseq.  $(a_{n_k})$  of  $(a_n)$  has a subseq.  
 $(a'_{n_k})$  with  $a'_{n_k} \rightarrow a$

Show then:  $\exists (f_n) \in L^\infty : f_n(x) \neq 0$  but  $\int f_n d\mu = 0$ . true -



Def 6.3 Def.  $S_n := \frac{1}{N} \sum_{n=0}^{N-1} T^n$  and.

$S^* f := \sup_{n \in \mathbb{N}} |S_n f|$   
 $S^*: L^1 \rightarrow \{\text{meas. fcts: } X \rightarrow \overline{\mathbb{R}}\}$   
 the maximal operator  
 corr. to  $(X, \mu, T)$

Rem. 1) A. l. conv. of  $\frac{1}{n} \sum_{k=1}^n$  is equiv. to conv. of  $\frac{1}{n} \sum_{k=0}^{n-1}$   
 (why?)

2) analog. one def.  $T^* f := \sup_n |T_n f|$  for any seq. of operators  
 on  $L^1(X, \mu)$ .  $T^*$  satisfies:

- $T^* f \geq 0 \quad \forall f$
- $T^*(2f) = |2| T^* f$
- $T^*(f+g) \leq T^* f + T^* g$  - subadditivity.

Def. 6.4 A seq.  $(T_n)$  of operators on  $L^1(X, \mu)$  satisfies a

maximal inequality if:

$$\mu(\{x : T^*f > \lambda\}) \leq C(\lambda)$$

for some  $f$  s.t.  $C : (0, \infty) \rightarrow [0, \infty)$  with

$$C(\lambda) \begin{cases} \rightarrow 0 \\ \rightarrow \infty \end{cases}$$

$$\begin{aligned} \lambda &> 0 \\ \forall f &\in L^1 \text{ with} \\ \|f\| &\leq 1 \end{aligned}$$

Thm. 6.5 (Banach's principle)

Let  $(T_n)$  be a seq. of lin. bdd operators on  $L^1(X, \mu)$  for a prob-space  $(X, \mu)$ . If the com. maximal operator satisfies a maximal ineq., then the set

$$F := \{f \in L^1 : (T_n f) \text{ conv. a.e.}\}$$

is a closed subspace of  $L^1(X, \mu)$

Proof subspace: clear:

Let  $f \in \overline{F}$  and def.

$$h := \lim_{k, l \rightarrow \infty} |T_k f - T_l f|$$

To show:  $h = 0$  a.e.

Let  $g \in F$ . Then:

$$\begin{aligned} |T_k f - T_l f| &\leq |T_k(f-g)| + |T_k g - T_l g| + |T_l(g-f)| \\ &\leq 2T^*(f-g) + |T_k g - T_l g| \end{aligned}$$

$$h \leq 2T^*(f-g)$$

max. ineq.

For  $\lambda > 0$  we have:

$$\mu[x: h > 2\lambda] \leq \mu[x: T^*(f-g) > \lambda] \stackrel{(1)}{\leq} C \left( \frac{\lambda}{\|f-g\|_1} \right) \xrightarrow[g \rightarrow F]{} 0,$$

so  $\mu[x : h > 2\lambda] = 0 \forall \lambda \Rightarrow h = 0$  a. s.

How to show max ineq.?

Thm 6.6 [Maximal ergodic thm]

For an MDS  $(X, \mu, T)$ , a real valued  $f \in L^1(X, \mu)$

and  $N \in \mathbb{N}$ ,

$$\int f(x) d\mu \geq 0$$

$$\left\{ x : f(x) + \dots + T^n f(x) > 0 \right\} \\ \text{for some } n \leq N$$

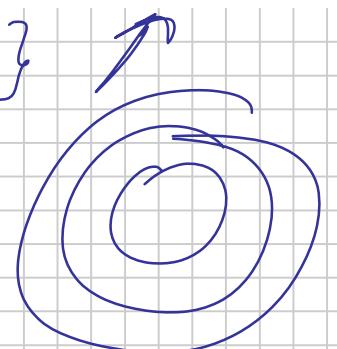
Th part.,  $\int f d\mu \geq 0.$

$$\left\{ x : f(x) + \dots + T^n f(x) > 0 \right\} \\ \text{for some } n$$

Proof "in part":  $E_N := \{x : f(x) + \dots + T^n f(x) > 0 \text{ for some } n \leq N\}$

$$\int_{E_N} f d\mu > 0 \quad \forall N \Rightarrow \int_{\bigcup_n E_N} f d\mu > 0$$

To show:  $\int_{E_N} f d\mu > 0$ .



For  $N \in \mathbb{N}$  def.:  $f_0 := 0$   
 $f_1 := f$   
 $f_n := f + Tf + \dots + T^{n-1}f$   
 $F_N := \max \{f_0, \dots, f_N\}$

We have:

- $F_N \geq 0$
- $F_N \geq f_n \quad \forall n = 0, \dots, N$ .

$$\text{So: } TF_n + f \geq T f_n + f = f + Tf + \dots + T^n f = f_{n+1} \quad \forall n \in \{0, \dots, N\}$$

In part.,  $TF_n + f \geq \max\{f_1, \dots, f_N\}$

$$\text{So: } TF_n + f \geq F_n \text{ on } E_N \quad (\text{here: } F_n > 0)$$

Because  $F_n > 0 \Rightarrow TF_n > 0$  we have:

$$\int_E f d\mu \geq \int_{E_N} F_n d\mu - \int_{E_N} TF_n d\mu \geq \int_X F_n d\mu - \int_X TF_n d\mu$$

$\uparrow X$   
 $F_n = 0 \text{ on } X \setminus E_N$   
 $\uparrow$   
This proves.

Corollary 6.7 (Maximal inequality)

For an MDS  $(X, \mu, T)$  and  $f \in L^1(X, \mu)$

$$\mu(\{x : (S^* f)(x) > \lambda\}) \leq \frac{\|f\|_1}{\lambda},$$

i.e.,  $S^*$  satisfy the max. ineq. for  $C(\lambda) := \frac{1}{\lambda}$ .

Proof: (HW) Hint: consider the max. erg. thm for  
the facts  $|f| - \lambda$

Now: max. ineq. 6.7 + Banach's principle b.5 + a.l. conv. on

$D$ -dense  $\Rightarrow$  pointwise erg. thm.

(HW) Show that for a uniquely ergodic system  
 $(X, T)$  with the unique inv. measure  $\mu$ :

$$\frac{1}{N} \sum_{n=1}^N (T^n f)(x) \rightarrow \int f d\mu \quad \forall f \in C(X)$$

(Hint: use ideas from the proof of Krylov-Bogolyubov)

Rem. 1) One can show using Birkhoff and ergodicity of the Bernoulli shift ( $Pf = \int f d\mu$ ) that a. l.  $x \in [0, 1]$  is normal, i. e.,  $x = 0.a_1 a_2 a_3 \dots$ ,  $a_i \in \{0, \dots, 9\}$  s. t.  $\forall d \quad \forall w_1, \dots, w_d \in \{0, \dots, 9\}$

$$\frac{\#\{j \in \{1, \dots, n\} : a_j = w_1, \dots, a_{j+d-1} = w_d\}}{n} \rightarrow \left(\frac{1}{10}\right)^d$$

(even & back),

pointwise erg. thm.

(Borel's thm, 1909)

2) One can show: PET  $\Rightarrow$  strong law of large numbers  
WET  $\Rightarrow$  weak —